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SU(4) instantons on Calabi-Yau threefolds with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group

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Abstract: Structure group SU(4) gauge vacua of both weakly and strongly coupled heterotic superstring theory compactified on torus-fibered Calabi-Yau threefolds Z with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group are presented. This is accomplished by constructing invariant, stable, holomorphic rank four vector bundles on the simply connected cover of Z. Such bundles can descend either to Hermite-Yang-Mills instantons on Z or to twisted gauge fields satisfying the Hermite-Yang-Mills equation corrected by a non-trivial flat B-field. It is shown that large families of such instantons satisfy the constraints imposed by particle physics phenomenology. The discrete parameter spaces of those families are presented, as well as a lower bound on the dimension of the continuous moduli of any such vacuum. In conjunction with $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson lines, these SU(4) gauge vacua can lead to standard-like models at low energy with an additional $\mathrm{U}(1)_{B-L}$ symmetry. This $\mathrm{U}(1)_{B-L}$ symmetry is very helpful in naturally suppressing nucleon decay.

Keywords: Superstrings and Heterotic Strings, M-Theory, Differential and Algebraic Geometry.

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1. Introduction

The compactification of both weakly coupled [1] and strongly coupled [2]–[4] heterotic string theory on smooth Calabi-Yau threefolds can potentially lead to phenomenologically interesting particle physics models in the four uncompactified dimensions. In both cases, the physical properties of the low energy theory are governed by the E_8 gauge field vacua on the internal space. These vacua determine the low energy gauge group and the number of quark and lepton families. In addition, they contribute moduli which play an important role in the structure of non-perturbative superpotentials [5, 6] and in vacuum stability [7]. It follows that constructing gauge field vacua, that is, instantons with structure group $G \subseteq E_8$ on Calabi-Yau threefolds, is of fundamental importance in the attempt to derive the real world from superstring theory.

It is well known how to construct G-instantons on four dimensional real manifolds and on complex surfaces. This is accomplished via the so-called ADHM construction [8]. This construction yields all information about the gauge connections and the associated holomorphic vector bundles. However, until relatively recently, little was known about how to construct arbitrary G-instantons on Calabi-Yau threefolds. This was due to several factors, not the least of which was the almost complete lack of knowledge about the metrics on such manifolds. This situation changed significantly with the work of [9, 10] and [11] where stable, holomorphic vector bundles with arbitrary structure group were constructed on simply connected elliptically fibered Calabi-Yau threefolds. These papers exploited the theorem of Uhlenbeck and Yau [12] and Donaldson [13] that proves that any such bundle must admit a unique gauge connection that satisfies the hermitean Yang-Mills equation. That is, finding G-instantons on an elliptically fibered Calabi-Yau threefold is equivalent to constructing stable holomorphic vector bundles with structure group G. It remains unknown how to solve the hermitean Yang-Mills equation, but [9, 10] and [11] open the door to constructing, relatively easily, the associated vector bundles.

These results were extended and applied to both weakly and strongly coupled heterotic superstring vacua in a series of papers. In [14]–[16], vector bundles were constructed on simply connected Calabi-Yau threefolds to produce low energy grand unified theories with gauge groups such as E_6, SO(10) and SU(5). This work was carried out within the context of compactified strongly coupled heterotic string theory, known as heterotic M-theory [17, 18]. Physically relevant phenomena such as $N = 1$ supersymmetry breaking [19, 20], five-brane moduli [21], non-perturbative superpotentials [22] and small instanton phase transitions [23] were explored in these theories. These concepts were also used to develop a new theory of the early universe, termed Ekpyrotic cosmology [24]. Although originally presented in heterotic M-theory, most of this work applies to compactifications of the weakly coupled heterotic string as well.

As constructed, these grand unified theories need non-zero vacuum expectation values to develop in the low energy theory in order to break to the standard model gauge symmetry. A more direct way to produce standard-like models is to allow Wilson lines. Mathematically this amounts to adding non-trivial flat bundles to the holomorphic vector bundles. However, this requires two new objects, torus-fibered Calabi-Yau threefolds with non-trivial fundamental group and stable, holomorphic vector bundles on such manifolds, to be constructed. The first steps in this direction were presented in a series of papers [25]–[28]. In these papers, torus-fibered Calabi-Yau threefolds Z with non-trivial $\mathbb{Z}_2$ fundamental group were constructed and relevant geometrical properties, such as the structure of $H_4(Z, \mathbb{Z})$, were discussed. Then, a general method for building stable, holomorphic vector bundles with structure group $G = \mathrm{SU}(5)$ on these manifolds was presented. The result was standard-like models at low energy, obtained by breaking the SU(5) low energy gauge group to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ with a $\mathbb{Z}_2$ Wilson line. These models, although promising, suffer from a potential deficiency. That is, there is no natural mechanism to suppress nucleon decay in this context. Although such suppression might be achieved, by discrete symmetries for example, it has to be implemented on a restricted, case by case basis.

This has prompted the authors to consider a major generalization of the work in [25]–[28] to include a much wider range of fundamental groups and structure groups. The question of nucleon decay has led us to construct, as the next step, torus-fibered Calabi-Yau threefolds with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group supporting stable, holomorphic vector bundles with $G = \mathrm{SU}(4)$ structure group. This choice of structure group was motivated by the results of [29, 30], where it was pointed out that such theories could lead to standard-like models with an additional $\mathrm{U}(1)_{B-L}$ gauge group. This $\mathrm{U}(1)_{B-L}$ symmetry, although not a complete mechanism for nucleon decay suppression, is very helpful in suppressing the most egregious dimension four operators. Also see [31]. In [32], we constructed torus-fibered Calabi-Yau threefolds Z with $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group over a base surface $d\mathbb{P}_9$. A detailed discussion of their geometry was given and their moduli spaces presented. An important ingredient in constructing standard-like models is the homology group $H_4(Z, \mathbb{Z})$. The explicit construction of this group in the present context is non-trivial. For this reason, we devoted a second paper [33] entirely to a discussion of $H_4(Z, \mathbb{Z})$ and its properties. However, there remains the fundamental issue of constructing $G = \mathrm{SU}(4)$ gauge instantons on these Calabi-Yau threefolds. That is explicitly accomplished in this paper.

Specifically, we do the following. In section 2, we review the results of [32] and [33] that will be required in our construction. We also use this opportunity to set our notation. Section 3 is devoted to a general discussion of rank four vector bundles in this context. First, bundles are constructed on elliptically fibered Calabi-Yau threefolds X with trivial fundamental group which admit a freely acting automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Any such bundle V that is, additionally, equivariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ will then descend to produce a stable, holomorphic rank four vector bundle V_Z on the torus-fibered Calabi-Yau threefold $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group. The Donaldson-Uhlenbeck-Yau theorem allows us to identify a unique connection of the bundle V_Z as a solution of the hermitean Yang-Mills equations on Z. Equivariance is a very strong constraint in this context. A weaker condition, which also leads to physical vacua, is the requirement that V only be invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$. For a detailed discussion of invariance versus equivariance see appendix A. Stable holomorphic bundles V on X which are invariant, but not equivariant, give rise to twisted vector bundles V_Z on Z, where the twisting is given by a flat B-field $\boldsymbol{B}$. See [34, 35, 36], section 3 and appendix A. Every such bundle comes equipped with a gauge field A which solves its string theory equation of motion. In the presence of a flat B-field $\boldsymbol{B}$, this equation is a version of the hermitean Yang-Mills equation in which the right hand side is modified by $\boldsymbol{B}$. That is, we have $F_A = \boldsymbol{B} \cdot \mathrm{id}$. See section 3 for details. Since the B-field is flat, we still obtain an $N = 1$ supersymmetric theory in the four dimensional low-energy limit. When the bundle V is actually equivariant, then the B-field is gauge equivalent to zero and we get back the usual hermitean Yang-Mills instantons. Mathematically, the invariant stable holomorphic bundles on X correspond to stable, holomorphic vector bundles on a suitably defined Deligne-Mumford gerbe over Z [36], but we will not discuss this interpretation here.

We outline the method we will use to produce invariant bundles, namely, by extensions from invariant rank two bundles V_i, $i = 1, 2$ on X. In each particular case, when we want to decide whether we are working in a trivial B-field background, we check the resulting bundles for equivariance as well. The constraints on V required by particle physics phenomenology, such as three families of quarks and leptons, are presented and discussed. It was shown, for example in [37], that any threefold X of the above structure must be a fiber product $X = B \times_{\mathbb{P}^1} B'$ of two rational elliptic surfaces B and B'. Each invariant rank two bundle V_i on X is explicitly composed from a rank two invariant bundle W_i on B and an invariant line bundle L_i on B'. The bundles W_i and L_i are constructed, their moduli spaces described, and their Chern character calculated in section 4 and section 5 respectively. Using this data, in section 6 rank two invariant vector bundles V_i are produced on X. These bundles are then used to construct $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant rank four vector bundles V on X by extension. In order for such vector bundles to admit a connection satisfying the hermitean Yang-Mills condition, it is necessary that they have the property of stability. The concept of stable holomorphic vector bundles is recalled in section 7 and a necessary condition for stability presented. A more complete, technical description of stability is given in appendix B. Section 8 is devoted to explicitly demonstrating that there is a large class of rank four vector bundles V on X that satisfy the standard model constraints and the necessary condition for stability. The discrete parameter space of these vector bundles is presented. In section 9 we prove that, subject to a mild restriction on this parameter space, all of these $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant holomorphic vector bundles are stable. This section relies on the technical discussion of stability given in appendix B. Finally, in section 10 we draw all of these results together into a conclusion. We present a large class of rank four, stable holomorphic vector bundles V on X that are $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant and explicitly give their discrete parameter space. We also use results derived throughout the paper to put a lower bound on the complex dimension of the continuous moduli space of any such bundle V and give a concrete example. As we have already explained, any such bundle V descends to a twisted hermitean Yang-Mills connection of standard model type on the torus-fibered Calabi-Yau threefold $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group, thus accomplishing our goal. In the process, we also analyze the conditions for the standard model vacuum to appear in a trivial B-field background. We find that, in our setting, the triviality of the B-field is incompatible with the three generation condition. Although it is likely that our method can be modified to produce standard model bundles with trivial B-field background, we will not pursue this in the present paper since the phenomenologically relevant constraints coming from the standard model are insensitive to this issue.

Admittedly, this paper is rather technical. This arises from the fact that one is, in principle, solving the very non-linear hermitean Yang-Mills equation on complicated Calabi-Yau threefolds whose metrics are unknown. From this point of view, it seems remarkable that this can be accomplished at all, let alone with the relatively straightforward mathematical techniques used in this paper. Be that as it may, the results of this paper are immediately relevant to pure particle physics, opening the door to the construction of standard-like models with suppressed nucleon decay within the context of heterotic superstring theory. In several upcoming papers, we extend our results to more general fundamental groups and structure groups and give a detailed analysis of the Wilson line symmetry breaking patterns [38]. We will also present standard-like models with $\mathrm{U}(1)_{B-L}$ symmetry [39] within this context.

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2. Calabi-Yau threefolds Z with $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group

Calabi-Yau threefolds Z with non-trivial first homotopy group

$$\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{2.1}$$

can be constructed as follows. Let X be a Calabi-Yau threefold with trivial fundamental group that admits a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms preserving the volume form. Then the quotient

$$Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2) \tag{2.2}$$

is a smooth Calabi-Yau threefold with first homotopy group (2.1). There are many types of Calabi-Yau threefolds X with this property. However, for the construction of standard-like model vacua in heterotic superstring theory, it is expedient [15] to choose X to be elliptically fibered. That is, X is fibered over a base B',

$$\pi : X \to B' , \tag{2.3}$$

with the generic fiber of π being isomorphic to a smooth torus T^2. Furthermore, this fibration admits a global section

$$\sigma : B' \to X , \tag{2.4}$$

which turns each generic fiber into an elliptic curve. The mapping σ is referred to as the zero section of X. As discussed in [32], choosing X to be elliptically fibered renders $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to be torus fibered, since Z does not admit a global section.

The base space B' of an elliptically fibered Calabi-Yau threefold is restricted to be either a del Pezzo surface, $d\mathbb{P}_r$ for $r = 1, \ldots, 8$, a Hirzebruch surface F_r for any non-negative integer r, an Enriques surface E, certain blow-ups of these surfaces or a rational elliptic surface $d\mathbb{P}_9$. However, for the purpose of constructing Calabi-Yau threefolds admitting freely acting automorphism groups with $\mathbb{Z}_2$ factors, it is particularly convenient [32] to choose

$$B' \cong d\mathbb{P}_9 . \tag{2.5}$$

The reason for this is the following. Recall that any rational elliptic surface is itself elliptically fibered over a $\mathbb{P}^1$ base with the projection map

$$\beta' : B' \to \mathbb{P}^{1'} . \tag{2.6}$$

Using this fact, it was shown, for example in [32], that any Calabi-Yau threefold X elliptically fibered over base (2.5) must be the fiber product over $\mathbb{P}^1$ of two rational elliptic surfaces B and B'. That is,

$$X = B \times_{\mathbb{P}^1} B' . \tag{2.7}$$

The fiber product permits the natural projections

$$
\begin{tikzcd}
 & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\
B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\
 & \mathbb{P}^1 &
\end{tikzcd}
\quad . \tag{2.8}
$$

For any rational elliptic surface B, it is relatively straightforward to construct all of the involutions

$$\tau_B : B \to B\,, \tag{2.9}$$

where $\tau_B^2 = id$. This was carried out in [32]. By appropriately restricting the moduli space of B, one can find surfaces that admit an involution and, hence, a $\mathbb{Z}_2$ automorphism group. Further restrictions of the moduli space allow several commuting involutions, leading to automorphism groups $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and so on. With this in mind, one chooses two rational elliptic surfaces B and $B^{'}$ each, for example, admitting a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Let τ_{Bi} and $\tau_{B'i}$ for $i = 1, 2$ be the associated generators. Then, as shown in [32], the Calabi-Yau threefold X constructed as the fiber product (2.7) of B and $B^{'}$ inherits a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by

$$\tau_{Xi} = \tau_{Bi} \times_{\mathbb{P}^1} \tau_{B'i}\,, \qquad i = 1, 2\,, \tag{2.10}$$

as desired. This explains our choice of $B^{'} \cong d\mathbb{P}_9$ as the base surface of X.

The explicit choice of the rational elliptic surfaces B and $B^{'}$ required to produce standard-like model vacua in heterotic superstring theory is rather subtle. In this paper, we are interested in producing realistic theories with $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson lines. Therefore, we must choose B and $B^{'}$ to each admit an automorphism group containing $\mathbb{Z}_2 \times \mathbb{Z}_2$. As discussed in [32], there exists a three parameter family of rational elliptic surfaces B that admit exactly $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism groups. However, as will become clear later in this paper, such surfaces do not generically lead to standard-like model vacua. To achieve such vacua, it is essential to further restrict B and $B^{'}$ to each be contained in a specific two-parameter sub-family of rational elliptic surfaces. This two parameter sub-family was described in detail in [32]. Here, we will simply recall the relevant properties. Let B be a $d\mathbb{P}_9$ surface in this two parameter sub-family. Then B necessarily has two properties shared by all rational elliptic surfaces. First, B is an elliptic fibration

$$\beta : B \to \mathbb{P}^1\,. \tag{2.11}$$

Second, the second homology group is given by

$$H_2(B, \mathbb{Z}) \cong \mathbb{Z}l \oplus \bigoplus_{i=1}^{9} e_i\,, \tag{2.12}$$

where l is the pull-back of a line in $\mathbb{P}^2$ and e_i, $i = 1, \ldots, 9$ are a set of divisors of B with self-intersection -1. These divisors each intersect every fiber of (2.11) only once. Note, however, that for B in the two parameter sub-family, not all of the e_i are sections of this fibration. More precisely, e_3, e_5 and e_8 each consist of two components. Hence, they are not irreducible and smooth, properties required of any section.

Generic $B \cong d\mathbb{P}_9$ have twelve singular I_1 fibers. However, as shown in [32], the restriction of B to the two parameter sub-family coalesces these fibers pairwise to produce six I_2 Kodaira fibers. These I_2 fibers are reducible, each being of the form

$$n_i \cup o_i \tag{2.13}$$

for $i = 1, \ldots, 6$. Another important consequence of restricting B to the two parameter sub-family is that the maximal number of commuting involutions is increased, from two in the three parameter case, to three. That is, B has a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by τ_{Bi}, $i = 1, 2, 3$. Be that as it may, only two generators,

$$\tau_{Bi}\,, \qquad i = 1, 2\,, \tag{2.14}$$

lift to a freely acting involution on X. Hence, we consider only the $\mathbb{Z}_2 \times \mathbb{Z}_2$ sub-group of automorphisms generated by (2.14). It was shown in [32] that these generators have the form

$$\tau_{B1} = t_{e_6} \circ \alpha_B\,, \qquad \tau_{B2} = t_{e_4} \circ \alpha_B\,, \tag{2.15}$$

where α_B is an involution on B that leaves the zero section $e_9 \equiv e$ invariant, e_4, e_6 are sections of B that intersect each fiber at a point of order two and t_{e_i}, $i = 1, 2$ are the translations corresponding to these sections. Note that

$$\tau_{B1} \circ \tau_{B2} = t_{e_4 + e_6} \tag{2.16}$$

which acts as a pure translation on the fibers.

A class in $H_2(B, \mathbb{Z})$ is called invariant if it is not transformed under the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Denote by

$$H_2(B, \mathbb{Z})^{\text{inv}} \subset H_2(B, \mathbb{Z}) \tag{2.17}$$

the homology sub-space of invariant classes. It was shown in [33] that for B in the two parameter sub-family

$$\text{rank} H_2(B, \mathbb{Z})^{\text{inv}} = 4\,. \tag{2.18}$$

Furthermore, a set of generators of $H_2(B, \mathbb{Z})^{\text{inv}} \otimes \mathbb{Q}$ is given by

$$i,\ f,\ n_1 + o_2,\ M, \tag{2.19}$$

where

$$\begin{aligned} i &= 2e_6 + 2e_4 - n_1 + o_2\,, \\ M &= 2e_2 - 2e_9 - n_1 - n_2\,, \end{aligned} \tag{2.20}$$

	i	f	$n_1 + o_2$	M
i	-4	4	4	0
f	4	0	0	0
$n_1 + o_2$	4	0	-4	0
M	0	0	0	-4

Table 1: The intersection numbers of the four generators of $H_2(B, \mathbb{Z})^{\mathrm{inv}} \otimes \mathbb{Q}$.

f is the fiber class of $\beta : B \to \mathbb{P}^1$ and n_1, n_2 and o_2 are components of some I_2 fibers. This choice of generators is motivated by their intersection numbers displayed in table 1. Note, in particular, that M is orthogonal to the three other classes. It can easily be seen that these four invariant generators do not generate $H_2(B, \mathbb{Z})^{\mathrm{inv}}$. For example,

$$\frac{1}{2}(i + M + f) = e_6 + e_4 + e_2 - e_9 - n_1 - n_2 + f \tag{2.21}$$

is an integral class in $H_2(B, \mathbb{Z})^{\mathrm{inv}}$. We would not have found this class if we had restricted the coefficients of $i, f, n_1 + o_2$ and M to be integers. We will have to deal with issues arising from this subtlety later in this paper. As discussed above, choosing both B and B' to be in the two parameter sub-family allows one to construct a Calabi-Yau threefold $X = B \times_{\mathbb{P}^1} B'$ that admits a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms. This is generated by the involutions

$$\tau_{Xi} = \tau_{Bi} \times_{\mathbb{P}^1} \tau_{B'i}\,, \qquad i = 1, 2\,, \tag{2.22}$$

where τ_{Bi} and $\tau_{B'i}$ for $i = 1, 2$ are involutions generating $\mathbb{Z}_2 \times \mathbb{Z}_2$ on B and B' respectively. Furthermore, as shown in detail in [33], X inherits a homology sub-group

$$H_4(X, \mathbb{Z})^{\mathrm{inv}} \subset H_4(X, \mathbb{Z})\,, \tag{2.23}$$

each of whose classes is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ and which has

$$\mathrm{rank} H_4(X, \mathbb{Z})^{\mathrm{inv}} = 7\,. \tag{2.24}$$

As we will see, both the rank and the structure of $H_4(X, \mathbb{Z})^{\mathrm{inv}}$, which derive from the properties of $H_2(B, \mathbb{Z})^{\mathrm{inv}}$, are required to achieve standard-like model vacua.

3. Rank four vector bundles on X and Z

Having constructed torus fibered Calabi-Yau threefolds Z with fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$, we begin our discussion of stable, twisted vector bundles with structure group $G = \mathrm{SU}(4)$ on Z. Note, that any rank four holomorphic vector bundle V_Z defined on $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ must descend from a rank four vector bundle V on X that is equivariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Therefore, we will produce vector bundles on Z by constructing equivariant vector bundles on X. More generally, we will consider twisted hermitean Yang-Mills vacua on Z, with the twisting being given by a flat B-field. Every

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such vacuum descends from a stable, holomorphic vector bundle on X which is $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, but not necessarily equivariant. Conversely, every stable invariant bundle on X descends to a twisted hermitean Yang-Mills instanton for some flat B-field on Z. See appendix A for a discussion. Geometrically, these vacua correspond to stable holomorphic bundles on a Deligne-Mumford gerbe on Z, which is determined by the B-field. Since the theory of such bundles is rather technical, we will not discuss it here. However, before proceeding with the construction, it will be instructive to give the general form of the twisted hermitean Yang-Mills fields as they appear in physics. Recall that a flat B-field in physics is understood as a locally defined closed two-form which transforms in a specific way under a group of generalized gauge transformations. Mathematically, flat B-fields are understood as connections on $\mathbb{C}^*$ gerbes on Z. These gerbes are parameterized, up to equivalence, by the cohomology group $H^2(Z, \mathbb{C}^*)$, and the actual B-fields are represented by cocycles for an appropriately chosen model of this cohomology group. This is analogous to the way a flat connection on a $\mathrm{U}(1)$ bundle is viewed as a cocycle representing a class in $H^1(Z, \mathrm{U}(1))$. Furthermore, if $\boldsymbol{B}$ is a flat B-field on Z and A is an hermitean connection on some $\boldsymbol{B}$-twisted vector bundle on Z, then the modified hermitean Yang-Mills equation for A becomes

$$F_A = \boldsymbol{B} \cdot I \,. \tag{3.1}$$

This is explained in greater detail in appendix A.

As mentioned above, it is expedient [15] to choose X to be elliptically fibered. The reason is that on such manifolds there exists a powerful tool, namely the spectral cover construction [9]–[11], for creating large families of holomorphic vector bundles. Furthermore, straightforward conditions on the spectral covers ensure stability of the corresponding vector bundles. It was shown in [11] that a sufficient condition for stability is simply that the spectral cover be smooth and irreducible. These conditions were easily fulfilled for vector bundles corresponding to grand unified theories [14, 15]. In these situations, the vector bundles are not required to be invariant under a non-trivial automorphism group. In [25, 26], vacua associated with standard-like models were constructed from rank five vector bundles. These bundles were required to be equivariant under a $\mathbb{Z}_2$ automorphism group on X. It was found that requiring $\mathbb{Z}_2$ equivariance greatly enhances the difficulty of producing stable, holomorphic bundles using spectral covers. Be that as it may, a family of $\mathbb{Z}_2$ equivariant stable, holomorphic vector bundles was so constructed.

In this paper, we want to produce standard-like model rank four, holomorphic vector bundles V that are stable and invariant under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on X. Our initial approach was to attempt to use the spectral cover construction. However, we found this to be untenable for the following reason. Recall that when X is elliptically fibered over a base $B' \cong d\mathbb{P}_9$, it must have the form $X = B \times_{\mathbb{P}^1} B'$. It was shown in [26] that, in this case, the spectral cover construction on X can be reduced to finding spectral covers on the rational elliptic surface B. The requirement that V be invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ puts strong constraints on the allowed spectral covers on B. We find that such spectral covers must be the union of multiples of the zero section $e : \mathbb{P}^1 \to B$ and the fiber f of B, not merely homologous to them. But such covers are highly singular and reducible. Therefore, there is no easy argument for the stability of the corresponding bundles. Hence, at this point, it seems more constructive to abandon the spectral cover approach and to build stable, rank four vector bundles directly. This conclusion would appear to nullify the reason for choosing X to be elliptically fibered. However, this is not the case. Although the fiber product structure $X = B \times_{\mathbb{P}^1} B'$ and the requirement that the bundles be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant makes it difficult to construct stable vector bundles via spectral covers, they greatly enable producing stable vector bundles as non-trivial extensions, as we now discuss.

In broad outline, we will proceed as follows. Stable rank four vector bundles V on X will be constructed as non-trivial extensions

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{3.2}$$

of rank two vector bundles V_i, $i = 1, 2$ on X. These are defined in terms of two rank two bundles W_i, $i = 1, 2$ on B and two line bundles L_i, $i = 1, 2$ on B'. Specifically,

$$V_i = \pi'^* W_i \otimes \pi^* L_i\,, \qquad i = 1, 2\,. \tag{3.3}$$

Vector bundles V can be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant even if one, or both, of V_1 and V_2 are not invariant. However, bundles V of this type are harder to construct and we will not discuss them in this paper. On the other hand, if V_1 and V_2 are both $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, then it is relatively straightforward to construct invariant vector bundles V defined via (3.2). Therefore, we will restrict the discussion in this paper to bundles V created by extension from $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant bundles V_1 and V_2. It follows from the definition of the rank two bundles V_1 and V_2 in (3.3) that they will be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant if the rank two bundles W_1, W_2 and the line bundles L_1, L_2 are invariant. In subsequent sections, we will present the construction of invariant bundles W_i, L_i and V_i for $i = 1, 2$ and explore their properties. Finally, it will be necessary to prove that non-trivial extensions V, created via (3.2) from V_1 and V_2, exist. This will be done in the appropriate places.

Before doing this, however, we want to impose certain restrictions on the (twisted or untwisted) holomorphic vector bundles V_Z and, hence, V that are required to produce standard-like model vacua. First, since we are interested in vector bundles with structure group $G = \mathrm{SU}(4)$, we must require that

$$c_1(V_Z) = 0\,. \tag{3.4}$$

Second, anomaly cancellation imposes the requirement that

$$c_2(TZ) - c_2(V_Z) = [W]\,, \tag{3.5}$$

where TZ is the tangent bundle of Z and $[W]$ represents the class of some holomorphic curve in Z around which five-branes are wrapped [15]. Note that (3.5) coincides with the anomaly cancellation condition in heterotic M-theory [2, 3] if one assumes that the E_8 vector bundle on the non-observable orbifold plane is trivial. When no five branes are present, condition (3.5) reduces to $c_2(V_Z) = c_2(TZ)$, which is difficult to solve. However, the existence of five-branes generalizes this to expression (3.5). At first view, (3.5) would appear to be simply a definition of $[W]$ and, hence, to put no constraint on $c_2(V_Z)$. However, this is not quite true. As discussed in [15], the five-brane class must be effective since it represents physical five-branes. Therefore, we require that

$$c_2(TZ) - c_2(V_Z) \qquad \text{effective}\,, \tag{3.6}$$

which puts a non-trivial constraint on $c_2(V_Z)$. Finally, in order for the low-energy theory to contain three families of quarks and leptons, one must impose [1]

$$c_3(V_Z) = 6\,. \tag{3.7}$$

Note that in writing these conditions, one needs to interpret the Chern classes of V_Z as cohomology classes on Z. In the untwisted case, V_Z is just a regular bundle and $c_i(V_Z)$ are the usual Chern classes which can be computed, for example, as the invariant polynomials of the field strength of any gauge field on V_Z. In the $\boldsymbol{B}$-twisted case, the curvature F_A of a twisted connection A is not a gauge invariant quantity. However, the expression $F_A - \boldsymbol{B}\cdot I$ is gauge invariant and should be understood as the appropriate field strength for the twisted gauge field A. The invariant polynomials of this expression define cohomology classes $c_i(V_Z)$ on Z. These are the Chern classes of the twisted bundle V_Z.

As discussed above, the (twisted or untwisted) bundle V_Z on Z will descend from a stable, rank four vector bundle V on X that is invariant under the automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since X is a cover of Z, it follows that $c_i(V)$ is the pull-back of $c_i(V_Z)$ for $i = 1, 2, 3$ in either case. Thus, on the level of cohomology classes on X, the difference between the twisted and the untwisted cases vanishes. Therefore, from (3.4), (3.6) and (3.7) we conclude that stable, rank four holomorphic vector bundles V on X will lead to standard-like models only if they obey the topological constraints

$$\begin{aligned} &(\mathbf{C1})c_1(V) = 0\,,\\ &(\mathbf{C2})c_2(TX) - c_2(V) \text{ effective}\,,\\ &(\mathbf{C3})c_3(V) = 24\,. \end{aligned} \tag{3.8}$$

These constraints will be imposed later in this paper.

4. Invariant rank two vector bundles on B

In this section, we construct rank two holomorphic vector bundles on B that are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by τ_{Bi}, $i = 1, 2$, and calculate their Chern characters. More precisely, we will construct the requisite rank two invariant vector bundles W_i, $i = 1, 2$ as non-trivial extensions of a rank one torsion free sheaf by a line bundle. We will find that the most natural invariant rank two bundles W are also, by construction, equivariant. Therefore, in the remainder of this paper, all bundles W will be assumed to have both properties.

To begin, let us consider two line bundles L_1 and L_2 on B. Their direct sum $L_1 \oplus L_2$ is a rank two vector bundle that fits into a short exact sequence

$$0 \longrightarrow L_1 \longrightarrow L_1 \oplus L_2 \longrightarrow L_2 \longrightarrow 0\ . \tag{4.1}$$

The direct sum corresponds to the trivial extension of L_2 by L_1. If $\{U_\alpha\}$ is an open cover of B, then the line bundles L_i, $i = 1, 2$ can be described in terms of their transition functions $g^i_{\alpha\beta}$, $i = 1, 2$ defined at each intersection $U_\alpha \cap U_\beta$. The data for $L_1 \oplus L_2$ on an intersection $U_\alpha \cap U_\beta$ is given by

$$g^{\oplus}_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & 0 \\ 0 & g^2_{\alpha\beta} \end{pmatrix} . \tag{4.2}$$

Unfortunately, as will be demonstrated later in this paper, rank two vector bundles of the form $L_1 \oplus L_2$ are too constrained to satisfy the conditions imposed by particle physics phenomenology. Therefore, we must consider generalizations of (4.2).

If one can modify $g^{\oplus}_{\alpha\beta}$ to

$$g^{\oplus}_{\alpha\beta} \to g^{'}_{\alpha\beta} = \begin{pmatrix} g^1_{\alpha\beta} & f_{\alpha\beta} \\ 0 & g^2_{\alpha\beta} \end{pmatrix} , \tag{4.3}$$

where $g^{'}_{\alpha\beta}$ fulfills the usual conditions on transition functions, then the new rank two holomorphic vector bundle W also fits into a short exact sequence of the form

$$0 \longrightarrow L_1 \longrightarrow W \longrightarrow L_2 \longrightarrow 0 \ . \tag{4.4}$$

Note that the zero in the lower left corner of (4.3) is necessary since (4.4) implies that L_1 is a sub-bundle of W. If $f_{\alpha\beta}$ is non-vanishing, then W is called a non-trivial extension of L_2 by L_1. We denote by

$$\mathrm{Ext}^1_B(L_2, L_1) \tag{4.5}$$

the set of all rank two holomorphic vector bundles W that fit into the exact sequence (4.4). Clearly, $\mathrm{Ext}^1_B(L_2, L_1)$ always contains the trivial extension $L_1 \oplus L_2$. In principal, $\mathrm{Ext}^1_B(L_2, L_1)$ can also contain non-trivial extensions W. $\mathrm{Ext}^1_B(L_2, L_1)$ can be shown to be a finite dimensional complex vector space. Since each vector space admits an abelian group structure, $\mathrm{Ext}^1_B(L_2, L_1)$ also has a group structure. We need discuss neither the vector space structure nor the group structure in any detail. Suffices it here to say that the trivial extension $L_1 \oplus L_2$ acts as the zero element of the vector space or, equivalently, as the identity element of the abelian group. Note that it is by no means obvious that any non-trivial extensions W exist. To determine if they do, one needs to calculate the vector space $\mathrm{Ext}^1_B(L_2, L_1)$ explicitly. To accomplish this, it is useful to note that

$$\mathrm{Ext}^1_B(L_2, L_1) \cong H^1(B, L_2^* \otimes L_1) , \tag{4.6}$$

where L_2^* denotes the dual of L_2. The cohomology group $H^1(B, L_2^* \otimes L_1)$ can usually be calculated.

However, as will become clear below, even non-trivial extensions satisfying (4.4) are not sufficient to satisfy the constraints of particle physics phenomenology. We must go one step further and consider extensions not of line bundles but, rather, of rank one torsion free sheaves by line bundles. Hence, we have to replace the line bundle L_2 in (4.4) by a torsion free sheaf. We will give a complete definition and discuss some properties of torsion free sheaves in the appendix B. For now, it is sufficient to know that every rank one torsion free sheaf on a smooth surface is of the form

$$L \otimes I_{z_k}\,, \tag{4.7}$$

where L denotes a line bundle on the surface and I_{z_k} the ideal sheaf of k points $z_k = \{p_1, \ldots, p_k\}$. Note that, in general, the points in z_k need not be isolated. However, in this paper, we will always restrict ourselves to ideal sheaves I_{z_k} where all of the points of z_k are distinct. To get an understanding of ideal sheaves, first recall that the structure sheaf $\mathcal{O}_B$ contains, for any open set $U \subset B$, all holomorphic functions $f : U \to \mathbb{C}$. Clearly, $\mathcal{O}_B$ is the trivial holomorphic line bundle on B, which explains the notation. Now consider the ideal sheaf I_p of a single point p on B. For any open set $U \subset B$ and $p \in U$, $I_p(U)$ contains all holomorphic functions $f : U \to \mathbb{C}$ which vanish on p. For any open set $U \subset B$ not containing p, $I_p(U)$ contains all holomorphic functions $f : U \to \mathbb{C}$. Hence, I_p is a sub-sheaf of the structure sheaf $\mathcal{O}_B$ of B. Therefore, we can write

$$0 \longrightarrow I_p \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_p \longrightarrow 0\,, \tag{4.8}$$

where $\mathcal{O}_p$ is the quotient sheaf

$$\mathcal{O}_p \cong \mathcal{O}_B / I_p\,. \tag{4.9}$$

This quotient sheaf is also called a “skyscraper” sheaf supported at p. Note that the fiber of $\mathcal{O}_p$ vanishes everywhere except at p, where it is isomorphic to $\mathbb{C}$. The ideal sheaf I_{z_k} of k distinct points can similarly be defined. Clearly I_{z_k} fits into the short exact sequence

$$0 \longrightarrow I_{z_k} \longrightarrow \mathcal{O}_B \longrightarrow \textstyle\bigoplus_{i=1}^k \mathcal{O}_{p_i} \longrightarrow 0\,, \tag{4.10}$$

where

$$\mathcal{O}_{p_i} \cong \mathcal{O}_B / I_{p_i} \tag{4.11}$$

for each p_i in z_k.

For any given line bundles L_1, L_2 and an ideal sheaf I_{z_k} on B, including the trivial sheaf $\mathcal{O}_B$, we now construct W as the extension of $L_2 \otimes I_{z_k}$ by L_1, that is

$$0 \longrightarrow L_1 \longrightarrow W \longrightarrow L_2 \otimes I_{z_k} \longrightarrow 0\,. \tag{4.12}$$

As above, we define

$$\mathrm{Ext}^1_B(L_2 \otimes I_{z_k}, L_1) \tag{4.13}$$

to be the set of all extensions fitting into the exact sequence (4.12). Again, $\mathrm{Ext}^1_B(L_2 \otimes I_{z_k}, L_1)$ is a complex vector space with the trivial extension $L_1 \oplus (L_2 \otimes I_{z_k})$ as the zero element. By construction, an element W of $\mathrm{Ext}^1_B(L_2 \otimes I_{z_k}, L_1)$ is a sheaf on B, but it is not necessarily a smooth vector bundle. The criterion for W to be a vector bundle is the following. First, note that a theorem by Serre [40] states that every rank two vector bundle on B is an element of $\mathrm{Ext}^1_B(L_2 \otimes I_{z_k}, L_1)$ for some L_1, L_2 and I_{z_k}. $W \in \mathrm{Ext}^1_B(L_2 \otimes I_{z_k}, L_1)$ is a rank two vector bundle if and only if W is locally free. For the general definition of locally free sheaves, we refer the reader to the appendix B. Here, we will give a criterion for finding locally free sheaves within the restricted context discussed below.

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In this paper, we will not attempt to solve (4.12) for arbitrary line bundles L_1 and L_2. Rather, we will restrict the line bundles L_1 and L_2 on B to be of the from

$$L_1 = \mathcal{O}_B(af)\,, \qquad L_2 = \mathcal{O}_B(bf) \tag{4.14}$$

where f denotes the fiber class of the elliptic fibration $\beta : B \to \mathbb{P}^1$ and $a, b \in \mathbb{Z}$. Furthermore, as indicated above, we will always choose the sheaves I_{z_k} to contain only distinct points. It follows that the extensions W must satisfy

$$0 \longrightarrow \mathcal{O}_B(af) \longrightarrow W \longrightarrow \mathcal{O}_B(bf) \otimes I_{z_k} \longrightarrow 0 \tag{4.15}$$

and form the vector space

$$\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))\,. \tag{4.16}$$

In addition, we will only be interested in elements W of this vector space that are rank two vector bundles, that is, are locally free. Unfortunately, not every extension of $\mathcal{O}_B(bf) \otimes I_{z_k}$ by $\mathcal{O}_B(af)$ is locally free. For example, the trivial extension

$$\mathcal{O}_B(af) \oplus (\mathcal{O}_B(bf) \otimes I_{z_k}) \tag{4.17}$$

is not locally free, as can easily be checked using the definition given in the appendix B. Therefore, it is essential that we ensure that there are elements in $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ which are locally free sheaves. In this context, the existence of locally free extensions can be established using the Cayley-Bacharach criterion [40]. First, this criterion implies that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af)) \geq 1\,. \tag{4.18}$$

Now, by definition, the generic elements form a dense, open subset of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$. Let K_B denote the canonical bundle of B. Then the Cayley-Bacharach criterion states that a generic element of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ is a locally free sheaf if and only if any global section

$$s \in H^0(B, \mathcal{O}_B(af)^* \otimes \mathcal{O}_B(bf) \otimes K_B) \tag{4.19}$$

which vanishes at $k-1$ points in z_k, vanishes at all points. Otherwise, there are no locally free extensions. Note that if there are no sections, or no section vanishes at $k-1$ points, then a generic element is locally free. Recall from [32] that for rational elliptic surfaces B,

$$K_B \cong \mathcal{O}_B(-f)\,. \tag{4.20}$$

Hence, to analyze the global sections in (4.19) we need to consider $H^0(B, \mathcal{O}_B(mf))$ for any integer m. Using $\beta : B \to \mathbb{P}^1$, it follows from

$$\mathcal{O}_B(mf) \cong \beta^* \mathcal{O}_{\mathbb{P}^1}(m) \tag{4.21}$$

and from the projection formula that

$$H^0\left(B, \mathcal{O}_B(mf)\right) = H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)\right)\,. \tag{4.22}$$

w	$\dim_{\mathbb{C}} \mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_{2r}}, \mathcal{O}_B(af))$
> 0	$w + 2r$
0	$2r$
-1	$2r - 1 + \max(1 - r, 0)$
≤ -2	$2r - 1 + \max(-w - r, 0)$

Table 2: The dimension of the the vector space $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_{2r}}, \mathcal{O}_B(af))$ for $a, b \in \mathbb{Z}, w \equiv a - b$ and $r \in \mathbb{Z}_{>0}$.

This implies that all global sections in $H^0(B, \mathcal{O}_B(mf))$ are pull-backs of global sections in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$. Thus, the sections in $H^0(B, \mathcal{O}_B(mf))$ are constant along any fiber of $\beta : B \to \mathbb{P}^1$. Therefore, all sections which vanish at a single point in a specific fiber, will vanish everywhere along that fiber. To ensure that the Cayley-Bacharach condition is satisfied, we will simply choose the points in the ideal sheaf I_{z_k} so that each fiber of $\beta : B \to \mathbb{P}^1$ which contains one of the points $p_i \in z_k$, contains at least one other such point. Then the global sections in $H^0(B, \mathcal{O}_B(af)^* \otimes \mathcal{O}_B(bf) \otimes K_B)$ obey the Cayley-Bacharach criterion and we have ensured that a generic element of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ corresponds to a locally free sheaf, that is, to a rank two vector bundle W.

Since the rank two vector bundles W defined via (4.15) play a vital role in our construction of rank four vector bundles V on X, let us be more precise about how many of these rank two vector bundles exist. Given any integers $a, b \in \mathbb{Z}$ and the ideal sheaf I_{z_k}, it is not too hard to calculate the dimension of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$. Let us assume that k is even, that is,

$$k = 2r \tag{4.23}$$

for any positive integer r. As we will show below, this choice is a necessary condition for the invariance of the rank two bundles under the automorphism group. Furthermore, to assure that the Cayley-Bacharach condition is satisfied, we will distribute the $2r$ points pairwise over r smooth fibers. Then, we can use the results of [42] where the dimension of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_{2r}}, \mathcal{O}_B(af))$ is calculated. We summarize the results in table 2. Note that these results depend only on

$$w \equiv a - b \in \mathbb{Z} \tag{4.24}$$

and the positive integer r. Then, for any fixed integers r, a and b, the rank two vector bundles W form a dense, open subset in $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_{2r}}, \mathcal{O}_B(af))$. For concreteness, let us consider two examples which will turn out to be useful later. First assume that

$$w = -2\,, \qquad r = 1\,. \tag{4.25}$$

Then, it follows from table 2 that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_2}, \mathcal{O}_B(af)) = 2. \tag{4.26}$$

The generic elements in $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_2}, \mathcal{O}_B(af))$ form, by definition, a dense open subset of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_2}, \mathcal{O}_B(af))$. Therefore, for $w = -2, r = 1$ we have a two dimensional moduli space of rank two vector bundles fitting into the short exact sequence (4.15).

A second example is the case

$$w = -4 \,, \qquad r = 4 \,. \tag{4.27}$$

It follows from table 2 that

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_8}, \mathcal{O}_B(af)) = 7 \tag{4.28}$$

and, hence, we obtain a seven dimensional moduli space of rank two vector bundles W satisfying (4.15). Having established that there are many rank two vector bundles on B, we now want to find those that are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generators τ_{B1} and τ_{B2}. Recall from (4.15) that the rank two bundles W are defined to be extensions of sheaves of the form $\mathcal{O}_B(bf) \otimes I_{z_k}$ by the line bundle $\mathcal{O}_B(af)$. It is possible for W to be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant even if either, or both, of $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf) \otimes I_{z_k}$ are not invariant. However, such bundles W are harder to construct and we will not consider them. In this paper, we will always choose both $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf) \otimes I_{z_k}$ to be explicitly $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. The consequence of this is that there will be a well-defined $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\operatorname{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$, as easily seen from the exact sequence (4.15). One can then use this action to determine the extensions W that are $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. Let us begin by giving the conditions for both $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf) \otimes I_{z_k}$ to be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant.

The invariance of line bundles of the form $\mathcal{O}_B(mf)$ for any integer m is easy to prove. First, note that the divisor class f is invariant under τ_{B1} and τ_{B2}, as shown in [33]. This implies that for any m the line bundle $\mathcal{O}_B(mf)$ is invariant and is our motivation for choosing L_1 and L_2 in (4.14) to be of the form $\mathcal{O}_B(mf)$. In fact, we can prove the equivariance of these line bundles. Recall from (4.21) that $\mathcal{O}_B(mf) \cong \beta^* \mathcal{O}_{\mathbb{P}^1}(m)$. Furthermore,

$$\beta \circ \tau_{Bi} = \tau_{\mathbb{P}^1} \circ \beta \,, \qquad i = 1, 2 \,, \tag{4.29}$$

where $\tau_{\mathbb{P}^1}$ denotes the induced involution on $\mathbb{P}^1$ which has the two fixed points located at 0 and ∞ [32]. Since every line bundle on $\mathbb{P}^1$ is equivariant with respect to the action of $\operatorname{Aut}(\mathbb{P}^1)$, it follows that $\mathcal{O}_{\mathbb{P}^1}(m)$ is $\tau_{\mathbb{P}^1}$ equivariant. But, using (4.21) and (4.29) we see that

$$\tau^*_{Bi} \mathcal{O}_B(mf) \cong \mathcal{O}_B(mf) \,, \tag{4.30}$$

thus proving the equivariance of line bundles of the form $\mathcal{O}_B(mf)$ under the involutions τ_{Bi}, $i = 1, 2$. It remains to check the equivariance of I_{z_k}. The ideal sheaf I_{z_k} will be equivariant if the set $z_k = \{p_1, \ldots, p_k\}$ is invariant under τ_{B1} and τ_{B2}. To achieve this, we need to make specific choices of z_k. First, consider a generic point p and define z_4 to be

$$z_4 = \{p, \tau_{B1}(p), \tau_{B2}(p), \tau_{B1} \circ \tau_{B2}(p)\}. \tag{4.31}$$

By construction, the four-tuples z_4 and, hence, the associated sheaves I_{z_4} are equivariant under the action of τ_{B1} and τ_{B2}. Second, let us choose any point q satisfying $q = \tau_{B1}(q)$. There are at least four such points in B. To see this, recall from [32] that α_B acts as $\alpha_B(a) = -a$ for any $a \in f_\infty$ and, hence, α_B leaves fixed all four points of order two on f_∞, including e_6. There are four points $q \in f_\infty$, with the property that

$$2q = e_6 \,. \tag{4.32}$$

Then

$$\tau_{B1}(q) = t_{e_6} \circ \alpha_B(q) = e_6 - q = q\,, \tag{4.33}$$

where we have used (2.15) and (4.32). Given any such q, we define

$$z_2 = \{q, \tau_{B2}(q)\}\,. \tag{4.34}$$

By construction, the two-tuples z_2 and the associated sheaves I_{z_2} are necessarily equivariant under τ_{B1} and τ_{B2}. Of course, we can also construct invariant two-tuples z_2, and the corresponding equivariant sheaves, by starting with a point $q^{'}$ that is fixed under τ_{B2}. The sheaves we will consider in this paper will be generated by a combination of these methods. For example, choose a general point p, some point $q \neq p$ satisfying $q = \tau_{B1}(q)$ and consider the six-tuple $z_6 = \{p, \tau_{B1}(p), \tau_{B2}(p), \tau_{B1} \circ \tau_{B2}(p), q, \tau_{B2}(q)\}$. Then, clearly, the associated ideal sheaf I_{z_6} is equivariant under τ_{B1} and τ_{B2}. Proceeding in this way, one can construct ideal sheaves with m four-tuples and n two-tuples, that is

$$I_{z_k}\,, \qquad k = 4m + 2n\,, \qquad m \in \mathbb{Z}_{\geq 0}\,, \qquad n = 0, \ldots, 4\,, \tag{4.35}$$

which are equivariant under τ_{Bi}, $i = 1, 2$. The fact that we restrict n to be $0 \leq n \leq 4$ reflects the following fact. Note that there are only four points $q^{'}$ on B which are fixed under τ_{B2} and lie on a smooth fiber. Since these four points get pairwise exchanged under τ_{B1}, there are only two different invariant two-tuples which can be constructed using points invariant under τ_{B2}. Considering also two-tuples which one obtains starting with a point invariant under τ_{B1} explains the bound. We will restrict the points in z_k to lie on smooth fibers. Then, for invariant k-tuples z_k, any fiber containing one point of z_k contains another such point as well. This follows simply from the fact that $\tau_{B1} \circ \tau_{B2}$ acts as a translation along the fibers of $\beta : B \to \mathbb{P}^1$. Thus, the Cayley-Bacharach criterion discussed above is automatically satisfied. Therefore, the existence of generic locally free extensions using these equivariant ideal sheaves I_{z_k} is ensured. Putting everything together, we conclude the following.

Choose $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf) \otimes I_{z_k}$ with

$$a, b \in \mathbb{Z}\,, \qquad k = 4m + 2n\,, \qquad m \in \mathbb{Z}_{\geq 0}\,, \qquad n = 0, \ldots, 4\,. \tag{4.36}$$

Then the vector space of extensions $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ contains a dense, open subset of rank two vector bundles W. Since $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf)$ are automatically equivariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$, and I_{z_k} is chosen to be equivariant under these automorphisms, it follows that both $\mathcal{O}_B(af)$ and $\mathcal{O}_B(bf) \otimes I_{z_k}$ are $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant. As a consequence, there is a natural action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$. Denote by

$$\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))^{\mathrm{inv}} \tag{4.37}$$

the subspace of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially. If such a subspace is non-trivial, then the $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant vector bundles W form a dense, open subset of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))^{\mathrm{inv}}$.

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We now proceed to show that if $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))$ is non-trivial then $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))^{\mathrm{inv}}$ not only is non-trivial, but its dimension can be made to satisfy the constraint

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))^{\mathrm{inv}}\geq\frac{1}{4}\dim_{\mathbb{C}}\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af)). \tag{4.38}$$

To show this, first note that the automorphism group $\mathbb{Z}_2\times\mathbb{Z}_2$ acts linearly on the vector space $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))$. Since $\mathbb{Z}_2\times\mathbb{Z}_2$ is abelian, its irreducible representations are one dimensional and we can diagonalize the action of $\mathbb{Z}_2\times\mathbb{Z}_2$ on $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))$. There are four different irreducible representations of $\mathbb{Z}_2\times\mathbb{Z}_2$ given by the four characters χ_c, $c=1,\ldots,4$. Hence, we have the decomposition

$$\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))=\oplus_{c=1}^4 n_c E_c\,, \tag{4.39}$$

where E_c is the one dimensional eigenspace transforming under the $c-th$ character χ_c and n_c is its multiplicity. If we choose the eigenspace $E_{c'}$ with the highest multiplicity $n_{c'}$, then we can modify the action of $\mathbb{Z}_2\times\mathbb{Z}_2$ on $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))$ so that this eigenspace corresponds to the trivial representation. It follows that

$$\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))^{\mathrm{inv}}\equiv n_{c'}E_{c'} \tag{4.40}$$

is non-trivial and, furthermore, that its dimension is bounded from below as in expression (4.38). As stated above, the equivariant vector bundles W form a dense, open subset of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))^{\mathrm{inv}}$.

Again, since these equivariant rank two bundles play a vital role in our construction, let us be even more precise about how many such vector bundles exist. Note that for the sheaves I_{z_k} in (4.35), one can always write $k=2r$ for some positive integer r. Furthermore, the points of z_k can always be pairwise distributed over $k/2$ smooth fibers. Therefore, we can use table 2 to calculate the dimension of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_k},\mathcal{O}_B(af))$. Let us again consider the two examples discussed earlier. In the first example, we chose

$$w=-2\,,\qquad r=1 \tag{4.41}$$

which implied (4.26) that

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_2},\mathcal{O}_B(af))=2\,. \tag{4.42}$$

It then follows from (4.38) that

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_2},\mathcal{O}_B(af))^{\mathrm{inv}}\geq 1\,. \tag{4.43}$$

Hence, in this case, we have at least a one dimensional moduli space of equivariant rank two vector bundles W. In the second example,

$$w=-4\,,\qquad r=4\,. \tag{4.44}$$

Then, we found in (4.28) that

$$\dim_{\mathbb{C}}\mathrm{Ext}^1_B(\mathcal{O}_B(bf)\otimes I_{z_8},\mathcal{O}_B(af))=7 \tag{4.45}$$

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and, hence,

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_8}, \mathcal{O}_B(af))^{\mathrm{inv}} \geq 2\,. \tag{4.46}$$

That is, in this example, we have at least a three dimensional moduli space of equivariant rank two vector bundles. And so on. Note that if we take an extension corresponding to a non-trivial character of $\mathbb{Z}_2 \times \mathbb{Z}_2$, then the corresponding vector bundle W will be automatically invariant but not necessarily equivariant. Of course, the dimension estimate above remains valid regardless of whether we work with invariant or equivariant bundles.

Having constructed equivariant rank two vector bundles W on B, we now proceed to calculate their Chern characters. Since the W fit into the short exact sequence

$$0 \longrightarrow \mathcal{O}_B(af) \longrightarrow W \longrightarrow \mathcal{O}_B(bf) \otimes I_{z_k} \longrightarrow 0\ , \tag{4.47}$$

it follows that their Chern characters are given by

$$\mathrm{ch}\, W = \mathrm{ch}\, \mathcal{O}_B(af) + \mathrm{ch}\, \mathcal{O}_B(bf) \cdot \mathrm{ch}\, I_{z_k}\,. \tag{4.48}$$

The definition of the Chern character of vector bundles and its interpretation in term of gauge invariant polynomials is standard and can be found, for example, in [43]. We note that for any line bundle L on a surface,

$$\mathrm{ch}\, L = 1 + c_1(L) + \frac{1}{2} c_1(L)^2\,. \tag{4.49}$$

Since $c_1(\mathcal{O}_B(mf)) = mf$ and $f^2 = 0$, we find for the Chern character of $\mathcal{O}_B(mf)$ that

$$\mathrm{ch}\, \mathcal{O}_B(mf) = 1 + mf\,. \tag{4.50}$$

However, the ideal sheaf I_{z_k} is not locally free and, hence, is not a vector bundle. Instead, it belongs to a larger class of sheaves on B, called coherent sheaves. How does one define the Chern character on coherent sheaves? The relevant point in this regard is that every coherent sheaf F on a smooth algebraic variety has a finite resolution by locally free sheaves. That is, there exists an exact sequence

$$0 \to V_n \to V_{n-1} \to \cdots \to V_0 \to F \to 0 \tag{4.51}$$

for some finite integer n such that all $V_i, i = 0, \ldots, n$ are vector bundles. One then defines $\mathrm{ch}\, F$ as

$$\mathrm{ch}\, F = \sum_{i=0}^{n} (-1)^i \mathrm{ch}\, V_i\,, \tag{4.52}$$

where $\mathrm{ch}\, V_i$ can be calculated using, for example, a connection on V_i. It can be shown that $\mathrm{ch}\, F$ does not depend on the chosen resolution and, hence, is well-defined. This gives a definition for $\mathrm{ch}\, I_{z_k}$. To actually calculate $\mathrm{ch}\, I_{z_k}$, however, it is more convenient to use the Grothendieck-Riemann-Roch theorem [44]. This theorem states the following. Let X and Y be smooth algebraic varieties and $h : X \to Y$ a smooth, proper mapping between them with the property that $h^{-1}(y)$ contains only a finite number of points for every $y \in Y$.

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Then, for any coherent sheaf F on X there is a relation between the Chern character ch F on X and the Chern character of its push forward h_*F on Y. This relation is given by

$$\text{ch } h_*F = h_*(\text{ch } F \cdot \text{td } X) \cdot \text{td }^{-1}Y\,. \tag{4.53}$$

Here td X and td Y denote the Todd classes of X and Y respectively. Loosely speaking, the Grothendieck-Riemann-Roch theorem fixes the "non-commutativity" of ch $\cdot\, h_*$ and $h_* \cdot$ch . Let us first apply this theorem to the calculation of the Chern character of an ideal sheaf I_p of a point. Recall from (4.8) that I_p fits into the short exact sequence

$$0 \longrightarrow I_p \longrightarrow \mathcal{O}_B \longrightarrow i_*\mathcal{O}_p \longrightarrow 0\ , \tag{4.54}$$

where $i : p \to B$ is the inclusion. Hence, its Chern character is given by

$$\text{ch } I_p = \text{ch } \mathcal{O}_B - \text{ch } i_*\mathcal{O}_p\,. \tag{4.55}$$

One can now use the Grothendieck-Riemann-Roch theorem to calculate ch $i_*\mathcal{O}_p$. To do this, take $X = p$, $Y = B$ and h to be the inclusion map

$$i : p \to B\,. \tag{4.56}$$

Therefore, using (4.53) and (4.56) we find

$$\text{ch } \mathcal{O}_p = \text{ch } i_*\mathcal{O}_p = i_*(\text{ch } \mathcal{O}_p \cdot \text{td } p) \cdot \text{td }^{-1}B\,. \tag{4.57}$$

Since $\mathcal{O}_p$ is the trivial bundle on p, its Chern character is simply unity. Similarly, since the tangent bundle of a point is trivial, we obtain

$$\text{td }(p) = 1\,. \tag{4.58}$$

The inclusion $i_*(1)$ corresponds to a point in B, namely, to the image $i(p)$. The intersection of $i(p)$ with td ^{-1}B vanishes for all classes in td ^{-1}B except for 1. This intersection clearly corresponds to a point in B. Therefore, it follows from these results and (4.57) that

$$\text{ch } \mathcal{O}_p = \text{pt}\,. \tag{4.59}$$

Using this result, and the fact that

$$\text{ch } \mathcal{O}_B = 1\,, \tag{4.60}$$

it follows from (4.55) that

$$\text{ch } I_p = 1 - \text{pt}\,. \tag{4.61}$$

For an ideal sheaf I_{z_k} with k distinct points, this result easily generalizes to

$$\text{ch } I_{z_k} = 1 - k\text{pt}\,. \tag{4.62}$$

Using this, (4.48) and (4.50), we find that the Chern character of W is given by

$$\begin{aligned}\text{ch } W &= 1 + af + (1 + bf)(1 - k\text{pt})\\ &= 2 + (a + b)f - k\text{pt}\,.\end{aligned} \tag{4.63}$$

To conclude, to construct a rank four bundle V on X we need to produce two rank two bundles on B. We denote these as W_i, $i = 1, 2$, and define them to be locally free extensions satisfying

$$0 \longrightarrow \mathcal{O}_B(a_i f) \longrightarrow W_i \longrightarrow \mathcal{O}_B(b_i f) \otimes I_{z_{k_i}} \longrightarrow 0 \ , \qquad i = 1, 2 \tag{4.64}$$

where $a_i, b_i \in \mathbb{Z}$. If, in addition, the ideal sheaves are chosen so that $k_i = 4m_i + 2n_i$ where $m_i \in \mathbb{Z}_{\geq 0}$, $n_i = 0, \ldots, 8$ then, by the above discussion, for each $i = 1, 2$ there exists a subspace $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))^{\mathrm{inv}}$ of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))$ of complex dimension at least one on which $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts trivially. Any generic element of $\mathrm{Ext}^1_B(\mathcal{O}_B(bf) \otimes I_{z_k}, \mathcal{O}_B(af))^{\mathrm{inv}}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant rank two vector bundle on B. The associated Chern characters are given by

$$\mathrm{ch}\, W_i = 2 + (a_i + b_i) f - k_i \mathrm{pt} \, , \qquad i = 1, 2 \, . \tag{4.65}$$

In the next section, we proceed to calculate equivariant line bundles on the surface B'.

5. Invariant line bundles on B'

Denote by $Pic(B')$ the space of all line bundles on the surface B' up to isomorphism. Consider the map

$$c_1 : Pic(B') \to H^2(B', \mathbb{Z}) \, , \tag{5.1}$$

where $c_1(L)$ is the first Chern class of any $L \in Pic(B')$. Since B' is simply connected, the mapping (5.1) is injective. Furthermore, since B' is a rational elliptic surface this map is also surjective. Therefore, c_1 determines an isomorphism

$$Pic(B') \cong H^2(B', \mathbb{Z}) \, . \tag{5.2}$$

Using Poincare duality, we know that $H^2(B', \mathbb{Z}) \cong H_2(B', \mathbb{Z})$ and, hence,

$$Pic(B') \cong H_2(B', \mathbb{Z}) \, . \tag{5.3}$$

It follows that every class in $H_2(B', \mathbb{Z})$ determines a line bundle on B' up to isomorphism. If we denote by $c_1(L)$ both the first Chern class of L and its Poincare dual, then (5.3) implies that any line bundle $L \in Pic(B')$ is uniquely determined by $c_1(L) \in H_2(B', \mathbb{Z})$. Therefore, to describe all line bundles on B', we must find an integral basis of $H_2(B', \mathbb{Z})$. Such a basis was given in section 2, where we established that

$$H_2(B', \mathbb{Z}) \cong \mathbb{Z} l' \oplus \bigoplus_{i=1}^{9} e_i' \, , \tag{5.4}$$

with l' being the pull-back of a line in $\mathbb{P}^2$ and e_i', $i = 1, \ldots, 9$ being divisors on B' with self-intersection -1 which intersect each fiber of $\beta' : B' \to \mathbb{P}^1$ only once.

Therefore finding all line bundles on B' which are invariant under $\tau_{B'1}$ and $\tau_{B'2}$, is equivalent to finding all classes in $H_2(B', \mathbb{Z})$ which are invariant under these involutions. The set of all invariant classes is denoted by $H_2(B', \mathbb{Z})^{\mathrm{inv}} \subset H_2(B', \mathbb{Z})$. As discussed in section 2, for $B^{'}$ in the two parameter sub-family,

$$\text{rank} H_2(B^{'}, \mathbb{Z})^{\text{inv}} = 4 \,. \tag{5.5}$$

Furthermore, a set of four generators of $H_2(B^{'}, \mathbb{Z})^{\text{inv}} \otimes \mathbb{Q}$ is given by

$$i^{'},\ f^{'},\ n_1^{'} + o_2^{'},\ M^{'}\ , \tag{5.6}$$

where

$$\begin{aligned} i^{'} &= 2e_6^{'} + 2e_4^{'} - n_1^{'} + o_2^{'} \,, \\ M^{'} &= 2(e_2^{'} - e_9^{'}) - n_1^{'} - n_2^{'} \,, \end{aligned} \tag{5.7}$$

$f^{'}$ denotes the fiber class of $\beta^{'} : B \to \mathbb{P}^1$ and $n_1^{'}, n_2^{'}$ and $o_2^{'}$ are components of some I_2 fibers. Since these generators span $H_2(B^{'}, \mathbb{Z})^{\text{inv}} \otimes \mathbb{Q}$, it follows that the most general form of a class in $H_2(B^{'}, \mathbb{Z})^{\text{inv}}$ is given by

$$\tilde{a}i^{'} + xf^{'} + y\left(n_1^{'} + o_2^{'}\right) + \bar{a}M^{'} \tag{5.8}$$

for $\tilde{a}, x, y, \bar{a} \in \mathbb{Q}$ such that (5.8) expanded in the basis (5.4) has integral coefficients. It follows from (5.2), (5.3) and (5.8) that $L \in Pic(B^{'})$ is invariant under $\tau_{B^{'}i}, i = 1, 2$ if and only if $c_1(L) \in H_2(B^{'}, \mathbb{Z})^{\text{inv}}$, that is,

$$c_1(L) = \tilde{a}i^{'} + xf^{'} + y\left(n_1^{'} + o_2^{'}\right) + \bar{a}M^{'} \,. \tag{5.9}$$

As explained above, this expression uniquely determines the line bundle L. The Chern character of L can be written as

$$\text{ch}\, L = 1_{B'} + c_1(L) + \frac{1}{2}c_1(L)^2 \,. \tag{5.10}$$

In fact, it is easy to show that the line bundles $\mathcal{O}_{B'}(f')$, $\mathcal{O}_{B'}(n_1^{'} + o_2^{'})$, $\mathcal{O}_{B'}(i')$ and $\mathcal{O}_{B'}(M')$ are actually $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant. Indeed, we already checked this for $\mathcal{O}_{B'}(f')$. To show the equivariance of $\mathcal{O}_{B'}(n_1^{'} + o_2^{'})$, we need only notice that the effective divisor $n_1^{'} + o_2^{'}$ is a pull-back of a curve in the quotient surface $B'/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Finally, for the equivariance of $\mathcal{O}_{B'}(i')$ and $\mathcal{O}_{B'}(M')$ we observe that, modulo f' and $n_1^{'} + o_2^{'}$, the divisors i' and M' are divisible by two. Thus, modulo equivariant line bundles, the bundles $\mathcal{O}_{B'}(i')$ and $\mathcal{O}_{B'}(M')$ are tensor squares of invariant line bundles. Since the obstruction for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant bundle to be equivariant is classified by an element in $\mathbb{Z}_2$, see appendix A, this implies the desired equivariance of $\mathcal{O}_{B'}(i')$ and $\mathcal{O}_{B'}(M')$.

As discussed previously, to construct rank four vector bundles V on X, we will need to choose two invariant line bundles L_i, $i = 1, 2$ on $B^{'}$. We do this by specifying

$$c_1(L_i) = \tilde{a}_i i^{'} + x_i f^{'} + y_i\left(n_1^{'} + o_2^{'}\right) + \bar{a}_i M^{'} \,, \qquad i = 1, 2 \tag{5.11}$$

with $\tilde{a}_i, x_i, y_i, \bar{a}_i \in \mathbb{Q}$ chosen so that $c_1(L_i)$, $i = 1, 2$ are integral classes. It is important to note that, even through the bundles $\mathcal{O}_{B'}(i')$, $\mathcal{O}_{B'}(f')$, $\mathcal{O}_{B'}(n_1^{'} + o_2^{'})$ and $\mathcal{O}_{B'}(M')$ are individually equivariant, this is not necessarily a property of the linear combination (5.11). A line bundle L_i will be equivariant if each of the coefficients $\tilde{a}_i, x_i, y_i, \bar{a}_i$ are integers. However, if any of these coefficients is rational then L_i, although invariant by construction, need not be equivariant. In fact, the standard-like model conditions will, in this paper, require that L_i be invariant, but not equivariant.

6. Invariant rank four vector bundles on X

In this section, we will construct rank four holomorphic vector bundles V on X that are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by $\tau_{Xi}, i = 1, 2$ and calculate their Chern character. We will do this using a method similar to that employed to construct the rank two vector bundles W on B in section 4. That is, we will construct V by extension.

To begin with, let us consider any pair of rank two holomorphic vector bundles V_1 and V_2 on X. Then, we define a rank four holomorphic vector bundle V by the exact sequence

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \ . \tag{6.1}$$

Denote by

$$\mathrm{Ext}^1_X(V_2, V_1) \tag{6.2}$$

the set of all rank four holomorphic vector bundles V that fit into the exact sequence (6.1). $\mathrm{Ext}^1_X(V_2, V_1)$ always contains the trivial extension $V_1 \oplus V_2$. In principle, $\mathrm{Ext}^1_X(V_2, V_1)$ can also contain non-trivial extensions V. As discussed previously, $\mathrm{Ext}^1_X(V_2, V_1)$ can be shown to be a finite dimensional vector space. In particular, it has the structure of an abelian group. The trivial extension $V_1 \oplus V_2$ acts as the zero element of the vector space or, equivalently, as the identity element of the abelian group. Note that it is by no means obvious that any non-trivial extensions V exists. To determine if they do, one needs to calculate the vector space $\mathrm{Ext}^1_X(V_2, V_1)$ explicitly. The techniques required for doing this go beyond those considered in this paper. Suffices it here to say that for the rank two bundles V_1 and V_2 that we will consider in this paper, we find that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1) \geq 1 \,. \tag{6.3}$$

That is, the sequence (6.1) always has non-trivial extensions. This is proven in [42]. Note that in section 4, the rank two extensions W were torsion free sheaves. Only a dense, open subset of these were, additionally, locally free and, hence, smooth vector bundles. Here, we do not have this problem. All elements of $\mathrm{Ext}^1_X(V_2, V_1)$ are holomorphic vector bundles.

Having shown that there are many rank four holomorphic vector bundles on X, we now want to find those that are invariant and, possibly, equivariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generators τ_{X1} and τ_{X2}. Henceforth, we will choose V_1 and V_2 to be either simultaneously invariant, but not equivariant, or simultaneously $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant. From the defining sequence (6.1) for V and the fact that the obstruction to $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariance is an element in $\mathbb{Z}_2$, see appendix A, it follows that there is a group action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathrm{Ext}^1_X(V_2, V_1)$. As before, the space $\mathrm{Ext}^1_X(V_2, V_1)$ will split into eigenspaces labeled by the characters of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and every element in such an eigenspace will give us an extension which is invariant as a bundle. The elements in the subspace

$$\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \tag{6.4}$$

of $\mathrm{Ext}^1_X(V_2, V_1)$ which carries the trivial representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ will correspond to invariant bundles. By an argument similar to that leading to equation (4.38), one can show that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \geq \frac{1}{4} \dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1) \,. \tag{6.5}$$

It follows from (6.3) that, for the rank two $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant bundles V_1 and V_2 that we will consider in this paper, there always exists a vector subspace of rank four, $\mathbb{Z}_2\times\mathbb{Z}_2$ invariant bundles V with

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \geq 1\,. \tag{6.6}$$

We now turn to a more explicit construction of these bundles. Again, if V_1 and V_2 happen, in addition, to be equivariant we will get a non trivial equivariant extension V.

Specifically, we will use the invariant rank two vector bundles on B and the invariant line bundles on $B^{'}$ to produce rank four vector bundles V on X which are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by τ_{Xi}, $i = 1, 2$. We begin by constructing invariant rank two vector bundles $\mathcal{V}$ on X. Define

$$\mathcal{V} = \pi^{'*}W \otimes \pi^* L\,, \tag{6.7}$$

where W is any equivariant rank two vector bundle on B discussed in section 4 and L is any invariant line bundle on $B^{'}$ presented in section 5. By construction, $\mathcal{V}$ is a rank two holomorphic vector bundle on X. We now note that it is also invariant under the action τ_{Xi}, $i = 1, 2$. First, consider the $\pi^{'*}W$ factor. By assumption, W is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group of the surface B. That is

$$\tau^*_{Bi} W \cong W\,, \qquad i = 1, 2\,. \tag{6.8}$$

Recall that the bundles W that we use are, in addition, equivariant. What about the pull-back bundle $\pi^{'*}W$ on X? Note that the action of each generator τ_{Xi} on X induces the action τ_{Bi} on B. That is

$$\pi^{'} \circ \tau_{Xi} = \tau_{Bi} \circ \pi^{'}\,, \qquad i = 1, 2\,. \tag{6.9}$$

This follows directly from the definition of X as the fiber product of two rational elliptic surfaces B and $B^{'}$, and from the definition of the involutions τ_{Xi}, $i = 1, 2$ in terms of involutions τ_{Bi} and $\tau_{B'i}$, $i = 1, 2$ on B and $B^{'}$ respectively. Now define

$$f_i = \pi^{'} \circ \tau_{Xi}\,, \qquad i = 1, 2\,. \tag{6.10}$$

Clearly, each satisfies

$$f_i : X \to B\,. \tag{6.11}$$

We can use these mappings to pull-back vector bundles from B to X. From (6.9) and (6.10), we can write the pull-back maps f_i^* as

$$f_i^* = \tau^*_{Xi} \circ \pi^{'*} = \pi^{'*} \circ \tau^*_{Bi}\,, \qquad i = 1, 2\,. \tag{6.12}$$

Applying the maps f_i^* to the rank two vector bundle W on B, we find

$$f_i^* W = \tau^*_{Xi}(\pi^{'*}W) = \pi^{'*}(\tau^*_{Bi}W) = \pi^{'*}W\,, \qquad i = 1, 2\,. \tag{6.13}$$

Note that, in the last equality, we have used the fact that the rank two bundle W is invariant and, hence, satisfies (6.8). Equating the second and last terms in (6.13), it follows that

$$\tau^*_{Xi}(\pi^{'*}W) = \pi^{'*}W\,, \qquad i = 1, 2\,. \tag{6.14}$$

Therefore, the invariance of W on B under τ_{Bi}, $i = 1,2$ ensures the invariance of $\pi^{'*}W$ on X under τ_{Xi}, $i = 1,2$. In addition, it is easy to to show that since W is also equivariant, so is $\pi^{'*}W$.

Now consider the π^*L factor. By assumption, L is invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group of the surface $B^{'}$. That is

$$\tau^*_{B'i} L \cong L\,, \qquad i = 1,2\,. \tag{6.15}$$

What about the pull-back bundle π^*L on X? An argument similar to the above can be applied to show the invariance of π^*L under τ_{Xi}, $i = 1,2$. Here, one notes that the action of each τ_{Xi} on X induces the action of $\tau_{B'i}$ on $B^{'}$, namely

$$\pi \circ \tau_{Xi} = \tau_{B'i} \circ \pi\,, \qquad i = 1,2\,. \tag{6.16}$$

Applying the same reasoning as above and the fact that L is chosen to satisfy (6.15), we conclude that

$$\tau^*_{Xi}(\pi^*L) = \pi^*L\,, \qquad i = 1,2\,. \tag{6.17}$$

That is, the invariance of L on $B^{'}$ under $\tau_{B'i}$, $i = 1,2$ ensures the invariance of π^*L on X under τ_{Xi}, $i = 1,2$.

Note that this agrees with our results in [33]. There, it was shown that to ensure invariance under τ_{Xi}, $i = 1,2$ for any divisor on X of the form $\pi^*D^{'}$ and $\pi^{'*}D$, where D and $D^{'}$ are divisors on B and $B^{'}$ respectively, we need only to show the invariance of D and $D^{'}$ under τ_{Bi}, $i = 1,2$ and $\tau_{B'i}$, $i = 1,2$ respectively. If one associates, for example, the line bundle $\mathcal{O}_{B'}(D^{'})$ to the divisor $D^{'}$ on $B^{'}$, then the line bundle $\pi^*\mathcal{O}_{B'}(D^{'})$ corresponds to the divisor $\pi^*D^{'}$ on X. The invariance of $\mathcal{O}_{B'}(D^{'})$ and $D^{'}$ is now sufficient to ensure the invariance of $\pi^*\mathcal{O}_{B'}(D^{'})$ and $\pi^*D^{'}$.

Using (6.7), the fact that

$$\tau^*_{Xi}\mathcal{V} \cong \tau^*_{Xi}(\pi^{'*}W) \otimes \tau^*_{Xi}(\pi^*L)\,, \qquad i,j = 1,2 \tag{6.18}$$

and (6.14), (6.17), we conclude that

$$\tau^*_{Xi}\mathcal{V} \cong \mathcal{V}\,, \qquad i = 1,2\,. \tag{6.19}$$

That is, the rank two vector bundles defined in (6.7) are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group of X if they are constructed using the equivariant rank two bundles W and the invariant line bundles L defined in section 4 and section 5 respectively. If in addition L is chosen to be equivariant, then we will get an equivariant bundle $\mathcal{V}$.

Having produced invariant (respectively equivariant) rank two bundles $\mathcal{V}$ on X, we can proceed to construct rank four holomorphic vector bundles V on X which are invariant (respectively equivariant) under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group generated by τ_{Xi}, $i = 1,2$. To do this, consider any two invariant rank two vector bundles

$$V_i = \pi^{'*}W_i \otimes \pi^*L_i\,, \qquad i = 1,2\,. \tag{6.20}$$

Note from our previous discussion that the equivariant bundles W_i and, hence, V_i for $i = 1, 2$ have non-trivial moduli spaces. For each V_i, choose a specific point in its moduli space, which we also denote by V_i. For this specific choice, we then construct V as extensions satisfying

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{6.21}$$

and consider $\mathrm{Ext}^1_X(V_2, V_1)$. Then, it follows from the previous discussion that there exists a vector subspace

$$\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \subseteq \mathrm{Ext}^1_X(V_2, V_1) \tag{6.22}$$

of complex dimension at least one, each of whose elements is $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. That is, if $V \in \mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ then

$$\tau^*_{Xi} V \cong V \,, \qquad i = 1, 2 \,. \tag{6.23}$$

Therefore, any such bundle will descend to a twisted, rank four vector bundle V_Z on the non-simply connected Calabi-Yau threefold

$$Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2) \,. \tag{6.24}$$

Note that these conclusions hold for the choice of any points V_i, $i = 1, 2$ in the respective moduli spaces.

The definition of V as an extension of V_2 by V_1 makes the calculation of its Chern character straightforward. It follows from (6.21) that

$$\mathrm{ch}\ V = \mathrm{ch}\ V_1 + \mathrm{ch}\ V_2 \,. \tag{6.25}$$

Using (6.20), the Chern characters of V_i, $i = 1, 2$ can be evaluated. They are

$$\mathrm{ch}\ V_i = \mathrm{ch}\ \pi'^* W_i \cdot \mathrm{ch}\ \pi^*(L_i) = \pi'^* \mathrm{ch}\ W_i \cdot \pi^* \mathrm{ch}\ (L_i) \,. \tag{6.26}$$

From (4.63) and (5.11), we find that

$$\begin{aligned} \mathrm{ch}\ V_i &= \pi'^*(2 + (a_i + b_i) f - k_i \mathrm{pt}) \cdot \pi^* \left(1_{B'} + c_1(L_i) + \frac{1}{2} c_1(L_i)^2 \right) \\ &= 2 + \pi^* \left(2c_1(L_i) + (a_i + b_i) f' \right) + \left(c_1(L_i)^2 + (a_i + b_i)(c_1(L_i) \cdot f') \right) f \times \mathrm{pt}' - \\ &\quad - k_i \mathrm{pt} \times f' - k_i (c_1(L_i) \cdot f) \mathrm{pt} \,. \end{aligned} \tag{6.27}$$

The Chern character of V now follows easily from (6.25) and (6.27). It is given by

$$\begin{aligned} \mathrm{ch}\ V &= 4 + \pi^* \left(2c_1(L_1) + 2c_1(L_2) + (a_1 + a_2 + b_1 + b_2) f' \right) + \\ &\quad + \left(c_1(L_2)^2 + c_1(L_2)^2 + (a_1 + b_1)(c_1(L_1) \cdot f') + (a_2 + b_2) \left(c_1(L_2) \cdot f' \right) \right) f \times \mathrm{pt}' - \\ &\quad - (k_1 + k_2)\, \mathrm{pt} \times f' - \left(k_1 (c_1(L_1) \cdot f') + k_2 \left(c_1(L_2) \cdot f' \right) \right) \mathrm{pt} \,. \end{aligned} \tag{6.28}$$

This formula will turn out to be important when we try to find vector bundles that are compatible with the physical constraints discussed in section 3.

7. Stability of vector bundles V

Let X be a smooth projective algebraic variety with $\dim_{\mathbb{C}} X = 3$. A divisor $H \in H^2(X, \mathbb{Z})$ is said to be ample if

$$H \cdot C > 0\,, \qquad H^2 \cdot S > 0 \tag{7.1}$$

for all irreducible, effective curves $C \subset X$ and all irreducible, effective surfaces $S \subset X$ and if

$$H^3 > 0\,. \tag{7.2}$$

It can be shown that every projective algebraic variety admits at least one ample class. Now consider a holomorphic vector bundle or a torsion free sheaf V on X. For the definition of torsion free sheaves, we refer the reader to appendix B. The "slope" of V with respect to an ample class H, denoted by $\mu_H(V)$, is defined to be

$$\mu_H(V) = \frac{c_1(V) \cdot H^2}{\mathrm{rank} V}\,. \tag{7.3}$$

Then V is said to be stable if there exists an ample class H with respect to which

$$\mu_H(F) < \mu_H(V) \tag{7.4}$$

for every torsion free sub-sheaf F in V with

$$\mathrm{rank} F < \ \mathrm{rank} V\,. \tag{7.5}$$

The notion of stability of V is important for the following reason. In order for V to correspond to a physical vacuum in superstring theory, it must admit a connection that satisfies the hermitean Yang-Mills equation. It was shown by Donaldson [13] and Uhlenbeck and Yau [12] that this will be the case if and only if V is holomorphic and stable. All of the vector bundles and torsion free sheaves V discussed in this paper are holomorphic. However, if V is to be a physical vacuum, we must also demonstrate that it is stable.

In this paper, X is a Calabi-Yau threefold elliptically fibered over a base $B' \cong d\mathbb{P}_9$. It follows that X is a projective algebraic variety and, hence, at least one ample class H exists. Since X is a fiber product of B and B', it is relatively straightforward to construct a set of ample classes on X from the ample classes on B and B'. This is accomplished as follows. Let $h_0 \in H^2(B, \mathbb{Z})$ be an ample class on B and $h_0' \in H^2(B', \mathbb{Z})$ be an ample class on B'. The pull-backs, $\pi'^* h_0$ and $\pi^* h_0'$, are not ample classes in $H^2(X, \mathbb{Z})$. However, it is not hard to show that their sum

$$H_0 = \pi'^* h_0 + \pi^* h_0' \in H^2(X, \mathbb{Z}) \tag{7.6}$$

is, in fact, ample in X. Now consider a third ample class $h' \in H^2(B', \mathbb{Z})$. As stated above, $\pi^* h' \in H^2(X, \mathbb{Z})$ is not ample. However,

$$H' = \pi^* h' \in H^2(X, \mathbb{Z}) \tag{7.7}$$

is an effective class. This is sufficient for the sum

$$H = H_0 + nH' \in H^2(X, \mathbb{Z}) \tag{7.8}$$

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to be an ample class in X for any integer $n \in \mathbb{Z}_{\geq 0}$. Of course, this construction supposes that we can find ample classes on B and B'. However, since B and B' are projective algebraic varieties, such ample classes will indeed exist. Furthermore, there is an easy tool for checking if a divisor is ample on these surfaces. This is called the Nakai-Moishezon criterion [41] which states the following. A divisor h on a surface is ample if and only if

$$h^2 > 0\,, \qquad h \cdot C > 0 \tag{7.9}$$

for every irreducible, effective curve C on the surface. This gives an easy criterion for finding ample classes on B and B'.

There may be many ample classes in $H^2(X, \mathbb{Z})$. Why, then, do we consider those of the form (7.8)? The reason is the following. It was shown in [11] that vector bundles which are constructed using smooth, irreducible spectral covers on X are always stable with respect to some ample class of the form (7.8) for sufficiently large n. Intuitively, H_0 ensures the ampleness of H and nH' increases the volume of the fiber to ensure stability. The vector bundles V in this paper, however, are not constructed from spectral data. They can indeed be so constructed, but the associated spectral cover will be reducible and non-reduced. Be that as it may, checking stability with respect to some ample class H of the form (7.8) seems reasonable. This will be done in the next two sections where, in fact, we demonstrate the stability of V with respect to ample classes of the form (7.8).

Proving the stability of V with respect to H requires checking that condition (7.4) holds for all sub-sheaves F in V. This is technical and we will postpone a more complete calculation until section 9. Here, however, we present a simple criterion that is necessary in order for V to be stable. This criterion will be checked, along with the conditions for obtaining standard-like model vacua, in the next section. First, note that, in this paper, we are taking the structure group of V to be $G = \mathrm{SU}(4)$. It follows that $c_1(V) = 0$ and, hence,

$$\mu_H(V) = 0\,. \tag{7.10}$$

Therefore, V will be stable with respect to H if and only if

$$\mu_H(F) < 0 \tag{7.11}$$

for every sub-sheaf F of V satisfying (7.5). Second, although there may be many sub-sheaves of V, one of them is obvious and natural. Recall that V is constructed as the extension

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0\;. \tag{7.12}$$

It follows that V_1 is always a rank two sub-bundle of V. Therefore, the stability of V requires, among other things, that

$$\mu_H(V_1) < 0\,. \tag{7.13}$$

This constraint is by no means sufficient for the stability of V. However, it is very necessary and, as it turns out, very restrictive. It is helpful to check this condition first, as we will do in the next section, before demonstrating the complete stability of V. We will call (7.13) the stability condition on V. This criterion immediately explains why we must consider vector bundles V that are non-trivial extensions of V_2 by V_1. Let us assume that V is the trivial extension, that is

$$V \cong V_1 \oplus V_2 \,. \tag{7.14}$$

This would imply that both V_1 and V_2 are sub-bundles of V. Since the slope of V_1 is required to be negative by (7.13) and the slope of V is zero by (7.10), it follows that $\mu_H(V_2) > 0$, which violates (7.4). Hence, a vector bundle V of the form (7.14) can never be stable. However, if V is a non-trivial extension, V_2 is not a sub-bundle, and this difficulty is removed. Happily, as shown in the previous section, both $\mathrm{Ext}^1_X(V_2, V_1)$ and $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ have dimensions of at least one, and, hence, each contains non-trivial elements. If $\mathrm{Ext}^1(V_2, V_1)$ is trivial, the best we can do is to find V_1 and V_2 both with vanishing slope. Then $V_1 \oplus V_2$ is potentially poly-stable. Poly-stable bundles, however, correspond to solutions of the hermitean Yang-Mills connection which are reducible. We will not consider this possibility in this paper.

8. Numerical conditions and standard model bundles

In this section, we impose the physical conditions (**C1**), (**C2**), (**C3**) and the stability condition (7.13) on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, holomorphic, rank four vector bundles constructed in section 6. In fact, since none of these four conditions depends on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance of V, the results of this section are valid for any element of $\mathrm{Ext}^1_X(V_2, V_1)$. They apply to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant bundles in $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ by restriction.

8.1 Numerical conditions

First, recall that the Chern character of a rank four vector bundle on a threefold X can be expanded in terms of the Chern classes [44] of V as

$$\mathrm{ch}\, V = 4 + c_1(V) + \frac{1}{2}(c_1(V)^2 - 2c_2(V)) + \frac{1}{6}(c_1(V)^3 - 3c_1(V) \cdot c_2(V) + 3c_3(V)) \,. \tag{8.1}$$

Comparing this expression with (6.28), we see that

$$c_1(V) = \pi^* \left(2c_1(L_1) + 2c_1(L_2) + (a_1 + a_2 + b_1 + b_2) f' \right) . \tag{8.2}$$

Condition (**C1**) in (3.8) then requires that

$$(\#\mathbf{C1}) \qquad 2c_1(L_1) + 2c_1(L_2) = -(a_1 + a_2 + b_1 + b_2) f' \,. \tag{8.3}$$

After imposing this condition on V, it follows from (6.28), (8.1) and (8.3) that

$$\begin{aligned} c_2(V) = & - \left(c_1(L_2)^2 + c_1(L_2)^2 + (a_1 + b_1) c_1(L_1) \cdot f' + (a_2 + b_2) c_1(L_2) \cdot f' \right) f \times \mathrm{pt}' + \\ & + (k_1 + k_2) \mathrm{pt} \times f' \end{aligned} \tag{8.4}$$

and

$$c_3(V) = -2 \left(k_1 \left(c_1(L_1) \cdot f' \right) + k_2 \left(c_1(L_2) \cdot f' \right) \right) \mathrm{pt} \,. \tag{8.5}$$

Recall from [32] that the second Chern class of the tangent bundle of the Calabi-Yau threefold X is given by

$$c_2(TX) = 12\left(f \times \mathrm{pt}^{'} + \mathrm{pt} \times f^{'}\right). \tag{8.6}$$

Using this and (8.4), we see that condition **(C2)** in (3.8) requires that the class

$$\begin{aligned}
&12\left(f \times \mathrm{pt}^{'} + \mathrm{pt} \times f^{'}\right) + \\
&+ \left(c_1(L_1)^2 + c_1(L_2)^2 + (a_1+b_1)c_1(L_1)\cdot f^{'} + (a_2+b_2)c_1(L_2)\cdot f^{'}\right) f \times \mathrm{pt}^{'} - \\
&- (k_1+k_2)\mathrm{pt} \times f^{'}
\end{aligned} \tag{8.7}$$

be effective. This leads to two numerical constraints,

$$(\#\mathbf{C2}a) \qquad k_1 + k_2 \leq 12 \tag{8.8}$$

and

$$(\#\mathbf{C2}b) \qquad c_1(L_2)^2 + c_1(L_2)^2 + (a_1+b_1)c_1(L_1)\cdot f^{'} + (a_2+b_2)c_1\left(L_2 \cdot f^{'}\right) + 12 \geq 0\,. \tag{8.9}$$

The three family condition **(C3)** in (3.8) applied to (8.5) implies that

$$(\#\mathbf{C3}) \qquad k_1(c_1(L_1)\cdot f^{'}) + k_2(c_1(L_2)\cdot f^{'}) = -12\,. \tag{8.10}$$

To analyze the stability condition (7.13) numerically, we have to calculate $\mu_H(V_1)$. Recall from the definition of μ_H given in (7.3) that to calculate the slope of any vector bundle, one must first compute H^2. Assume that H is of the form (7.8). Then,

$$\begin{aligned}
H^2 &= \left(H_0 + nH^{'}\right)^2 \\
&= \left(\pi^{'*}h_0 + \pi^* h_0^{'}\right)^2 + 2n\left(\pi^{'*}h_0 + \pi^* h_0^{'}\right)\cdot \pi^* h^{'} + n^2 \pi^* h^{'2} \\
&= h_0^2\left(pt \times f^{'}\right) + 2\left(\pi^* h_0^{'} + n\pi^* h^{'}\right)\cdot \pi^{*'} h_0 + \pi^*(h_0^{'} + nh^{'})^2\,,
\end{aligned} \tag{8.11}$$

where we have used (7.6) and (7.7). Second, note that the first Chern class of the rank two bundle V_1 can be extracted from (6.27). It is given by

$$c_1(V_1) = \pi^*\left(2c_1(L_1) + (a_1+b_1)f^{'}\right). \tag{8.12}$$

Using this, expression (8.11), $\mathrm{rank}V_1 = 2$ and the facts that

$$\pi^*\left(2c_1(L_1) + (a_1+b_1)f^{'}\right)\cdot H^2 = \left(2c_1(L_1) + (a_1+b_1)f^{'}\right)\cdot \pi_* H^2 \tag{8.13}$$

and

$$\pi_*\pi^{'*}h_0 = h_0 \cdot f + p\mathrm{pt}\,, \qquad p \in \mathbb{Z}_{\geq 0}\,, \tag{8.14}$$

we find that

$$\begin{aligned}
\mu_H(V_1) = &\frac{1}{2}\left(2c_1(L_1) + (a_1+b_1)f^{'}\right)\cdot(h_0^2 f^{'} + 2h_0^{'}(h_0\cdot f)) + \\
&+ n\left(2c_1(L_1) + (a_1+b_1)f^{'}\right)\cdot h^{'}(h_0 \cdot f)\,.
\end{aligned} \tag{8.15}$$

Recall that we want to obtain stable bundles for large values of n. Clearly, for large n the term $n(2c_1(L_1)+(a_1+b_1)f^{'})\cdot h^{'}(h_0\cdot f)$ will be dominant. Furthermore, $h_0\cdot f>0$ by the ampleness assumption on h_0. Therefore, we find that the stability condition (7.13) requires that

$$\left(2c_1(L_1)+(a_1+b_1)f^{'}\right)\cdot h^{'}<0\,. \tag{8.16}$$

Before solving these constraints, we will make one arbitrary choice. Intersect (#**C1**) in (8.3) with $f^{'}$. Since $f^{'2}=0$, we find that

$$c_1(L_1)\cdot f^{'}+c_1(L_2)\cdot f^{'}=0\,. \tag{8.17}$$

Assuming $c_1(L_1)\cdot f^{'}=0$ would imply that $c_1(L_2)\cdot f^{'}=0$, in contradiction to (#**C3**) in (8.10). Therefore, we need to require that $c_1(L_1)\cdot f^{'}\neq 0$. Clearly, either

$$c_1(L_1)\cdot f^{'}>0 \tag{8.18}$$

and, hence, from (8.17) that

$$c_1(L_1)\cdot f^{'}>c_1(L_2)\cdot f^{'}\,, \tag{8.19}$$

or

$$c_1(L_1)\cdot f^{'}<0 \tag{8.20}$$

implying

$$c_1(L_1)\cdot f^{'}<c_1(L_2)\cdot f^{'}\,. \tag{8.21}$$

Henceforth, we will choose condition (8.19). This choice is motivated by the calculation in [26] where (8.19) is imposed to ensure the existence of non-trivial elements in $\mathrm{Ext}^1_X(V_2,V_1)$. We will show in [42] that this choice also ensures the existence of non-trivial elements of $\mathrm{Ext}^1_X(V_2,V_1)$ in the context of this paper. Choosing condition (8.21) would not allow us to arrive at this important conclusion.

In the next sub-section, we will solve all of the constraints (#**C1**), (#**C2**a), (#**C2**b), (#**C3**), (8.16) and (8.19) simultaneously, thus proving the existence of rank four vector bundles that are consistent with particle physics phenomenology.

8.2 Solution

In this sub-section, we will construct rank four vector bundles which solve all of the numerical conditions described above. We begin as follows. Note that (8.17) and (8.19) imply

$$0<c_1(L_1)\cdot f^{'}=-c_1(L_2)\cdot f^{'}\,. \tag{8.22}$$

Using this result in (#**C3**) gives

$$(k_2-k_1)c_1(L_1)\cdot f^{'}=12\,. \tag{8.23}$$

Let us define

$$k\equiv k_2-k_1\,. \tag{8.24}$$

k	k_1	k_2	k_1+k_2	$c_1(L_1)\cdot f'$	$c_1(L_2)\cdot f'$
2	0	2	2	6	-6
4	0	4	4	3	-3
6	0	6	6	2	-2
2	2	4	6	6	-6
4	2	6	8	3	-3
6	2	8	10	2	-2
2	4	6	10	6	-6
4	4	8	12	3	-3
12	0	12	12	1	-1

Table 3: Allowed values for k, k_1, k_2, $c_1(L_1)\cdot f'$, $c_1(L_2)\cdot f'$.

Since $c_1(L_1)\cdot f^{'}$ is positive, it follows from (8.23) that

$$k > 0 \tag{8.25}$$

and, hence,

$$k_1 < k_2\,. \tag{8.26}$$

Furthermore, since we will ensure that L_1 is a line bundle, the intersection $c_1(L_1)\cdot f^{'}$ is integral. It then follows from this and (8.23) that

$$k \text{ divides } 12\,. \tag{8.27}$$

Now, recall from (4.35), that k_1 and k_2 must be of the form

$$k_i = 4m_i + 2n_i\,, \qquad m_i \in \mathbb{Z}_{\geq 0}\,, \qquad n_i = 0,\ldots,8\,, \qquad i = 1,2\,. \tag{8.28}$$

Clearly, (8.25)–(8.28) and (#**C2**a) put severe restrictions on the allowed values of k, k_1 and k_2. In fact, there is only a small, finite set of solutions, which we list in the first three columns of table 3. The fourth column ensures that condition (#**C2**a) is obeyed. The fifth and sixth columns give the corresponding intersection numbers of $c_1(L_1)\cdot f^{'}$ and $c_1(L_2)\cdot f^{'}$ which solve (8.22) and (8.23).

To proceed, recall that L_i, $i=1,2$ must be invariant line bundles on $B^{'}$. It was shown in section 5 that, for this to be the case, their first Chern classes must satisfy

$$c_1(L_i) = \tilde{a}_i i^{'} + x_i f^{'} + y_i(n_1^{'} + o_2^{'}) + \bar{a}_i M^{'}\,, \qquad i = 1,2\,. \tag{8.29}$$

In the following calculation, we will make frequent use of the intersection numbers of the classes $i^{'}, f^{'}, n_1^{'} + o_2^{'}$ and $M^{'}$ which were given in table 1. The convenience of this choice of generators, with these specific intersection numbers, will now become clear. Intersecting $c_1(L_i)$, $i=1,2$ with $f^{'}$, and using (8.29), we find that

$$c_1(L_1)\cdot f^{'} = 4\tilde{a}_1\,, \qquad c_1(L_2)\cdot f^{'} = 4\tilde{a}_2\,. \tag{8.30}$$

Note that conditions (8.22) and (8.23) then fix the form of $\tilde{a}_1$ and $\tilde{a}_2$. We obtain

$$\tilde{a}_1 = \frac{3}{k}\,, \qquad \tilde{a}_2 = -\frac{3}{k}\,. \tag{8.31}$$

It follows that the line bundles L_i, $i = 1, 2$ are restricted to satisfy

$$c_1(L_1) = \frac{3}{k} i' + x_1 f' + y_1(n_1' + o_2') + \bar{a}_1 M' \tag{8.32}$$

and

$$c_1(L_2) = -\frac{3}{k} i' + x_2 f' + y_2(n_1' + o_2') + \bar{a}_2 M'\,. \tag{8.33}$$

Note that choosing k_1, k_2 and k from table 3 and line bundles satisfying (8.32) and (8.33) solves the constraints (#**C2**a), (#**C3**) and (8.19) completely. It also solves part of the condition (#**C1**). To complete (#**C1**), we have to impose the constraints

$$\begin{aligned} 2x_1 + 2x_2 &= -(a_1 + a_2 + b_1 + b_2)\,, \\ y_1 &= -y_2\,, \\ \bar{a}_1 &= -\bar{a}_2\,. \end{aligned} \tag{8.34}$$

If the coefficients x_i, y_i and $\bar{a}_i$ of $c_1(L_i)$, $i = 1, 2$ fulfill the conditions in (8.34), then the vector bundle V has vanishing first Chern class and, hence, solves (#**C1**).

To solve constraint (#**C2**b) in (8.9), we need to compute $c_1(L_i)^2$, $i = 1, 2$. Using the expressions (8.32)–(8.34) and table 1, we find that

$$c_1(L_1)^2 = -\frac{36}{k^2} - 4y_1^2 - 4\bar{a}_1^2 + \frac{24}{k}(x_1 + y_1) \tag{8.35}$$

and

$$c_1(L_2)^2 = -\frac{36}{k^2} - 4y_2^2 - 4\bar{a}_2^2 - \frac{24}{k}(x_2 + y_2)\,. \tag{8.36}$$

Constraint (#**C2**b) can now be written as

$$-\frac{72}{k^2} - 4y_1^2 - 4y_2^2 - 4(\bar{a}_1^2 + \bar{a}_2^2) + \frac{24}{k}(x_1 + y_1 - x_2 - y_2) + \frac{12}{k}(a_1 + b_1 - a_2 - b_2) \geq -12\,. \tag{8.37}$$

After implementing the conditions (8.34) and completing the squares, we find that

$$-8\left(y - \frac{3}{k}\right)^2 - 8\bar{a}^2 + \frac{24}{k}(x_1 - x_2) + \frac{24}{k}(a_1 + b_1 + x_1 + x_2) \geq -12\,, \tag{8.38}$$

where we have defined

$$y \equiv y_1 = -y_2\,, \qquad \bar{a} \equiv \bar{a}_1 = -\bar{a}_2\,. \tag{8.39}$$

Furthermore, defining

$$u \equiv 2y - \frac{6}{k}\,, \qquad x \equiv 2x_1 + a_1 + b_1 \tag{8.40}$$

allows us to simplify condition (#**C2**b) in (8.38) to

$$-u^2 - 4\bar{a}^2 + \frac{12}{k} x \geq -6\,. \tag{8.41}$$

The expressions (8.32) and (8.33) for $c_1(L_i)$, $i = 1, 2$ written in terms of x, u and $\bar{a}$ are

$$c_1(L_1) = \frac{3}{k}i' + \frac{1}{2}(x - a_1 - b_1)f' + \frac{1}{2}(u + \frac{6}{k})(n_1 + o_2) + \bar{a}M' \tag{8.42}$$

and

$$c_1(L_2) = -\frac{3}{k}i' + \frac{1}{2}(-x - a_2 - b_2)f' + \frac{1}{2}(-u - \frac{6}{k})(n_1 + o_2) - \bar{a}M' \,. \tag{8.43}$$

Note that if $u, \bar{a}$ and k obey the relations imposed by condition (8.41), then the vector bundle V, defined in terms of line bundles L_i, $i = 1, 2$ satisfying (8.42), solves both constraints (#**C1**) and (#**C2**b). Finally, note that if we use the expansion for $c_1(L_1)$ given in (8.42), then the stability condition (8.16) can be written as

$$(\frac{6}{k}i' + xf' + \left(u + \frac{6}{k})(n_1' + o_2'\right) + 2\bar{a}M) \cdot h' < 0 \,. \tag{8.44}$$

To complete our numerical solution, we have to find values of $u, \bar{a}, k, x, a_1, b_1, a_2, b_2$ for which equations (8.41) and (8.44) are satisfied, and such that the expressions $c_1(L_i)$, $i = 1, 2$ in (8.42) are integral. That is, such that L_i, $i = 1, 2$ are indeed line bundles. The solution we are presenting in this paper is specified by

$$u = -1 \,, \qquad \bar{a} = \frac{1}{2} \,, \qquad k = 6 \,, \qquad x = -1 \,, \qquad a_i + b_i \in 2\mathbb{Z} \,, \qquad i = 1, 2 \,. \tag{8.45}$$

Other, less obvious, solutions may exists. But we will not pursue them here. Using (8.31), (8.34), (8.39) and (8.40), we see that this solution corresponds to taking

$$\tilde{a}_1 = -\tilde{a}_2 = \frac{1}{2} \,, \qquad y_1 = y_2 = 0 \,, \qquad \bar{a}_1 = -\bar{a}_2 = \frac{1}{2} \tag{8.46}$$

and

$$x_1 = \frac{-a_1 - b_1 - 1}{2} \,, \qquad x_2 = \frac{-a_2 - b_2 + 1}{2} \tag{8.47}$$

with $a_i + b_i \in 2\mathbb{Z}$, $i = 1, 2$. In addition, we see from table 1 that $k = 6$ can occur in two ways, either

$$k_1 = 2 \,, \qquad k_2 = 8 \tag{8.48}$$

or

$$k_1 = 0 \,, \qquad k_2 = 6 \,. \tag{8.49}$$

Note, however, that (8.49) implies that V_1 is isomorphic to the sum of two line bundles. Therefore, vector bundles V corresponding to (8.49) are not irreducible. Hence, we discard them and allow (8.48) only. For the solution presented in (8.45), the left hand side of condition (8.41) reads

$$-u^2 - 4\bar{a}^2 + \frac{12}{k}x = -4 \geq -6 \,, \tag{8.50}$$

which satisfies this constraint. To proceed, it is convenient to define a divisor D' given by

$$D' \equiv \frac{1}{2}(i' - f' + 2\bar{a}M') = e_2' + e_4' + e_6' - e_9' - n_1' - n_2' \,. \tag{8.51}$$

Note that the first Chern class of the line bundles L_i, $i = 1, 2$ can be written in terms of this divisor. We find that

$$c_1(L_1) = D^{'} - \frac{1}{2}(a_1 + b_1)f^{'}, \qquad c_1(L_2) = -D^{'} - \frac{1}{2}(a_2 + b_2)f^{'}. \tag{8.52}$$

Since the divisor $D^{'}$ is clearly integral, the Chern classes defined in (8.52) are also integral for $a_i + b_i \in 2\mathbb{Z}$, $i = 1, 2$. Hence L_i, $i = 1, 2$ are line bundles, as required. By construction the line bundle $\mathcal{O}_{B'}(D')$ and, hence, the line bundles L_1 and L_2 are $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. However $\mathcal{O}_{B'}(D')$ can not be equivariant. Indeed, if it is, then its restriction to the fiber f_0', that is, the fiber $f^{'}$ over the point $0 \in \mathbb{P}^1$, will also be equivariant. On the other hand, the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ preserves f_0' and acts on it as translations by points of order two. But $D' \cdot f' = 2$. Therefore, $\mathcal{O}_{B'}(D')|_{f_0'}$ can not be a pull-back from $f_0'/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ since, in that case, $D' \cdot f'$ would have to be divisible by four.

Using (8.51), constraint (8.44) becomes the condition that

$$D^{'} \cdot h^{'} < 0 \tag{8.53}$$

for some ample class $h^{'}$ on $B^{'}$. To solve this condition, we must find such an ample class. The existence of this class is by no means obvious. For example, if $D^{'}$ were to be an effective divisor and, hence, consist of irreducible, effective curves in $B^{'}$, we would have $D^{'} \cdot h^{'} > 0$ for all ample classes $h^{'}$. This follows from the Nakai-Moishezon criterion discussed previously. Therefore, it is necessary to show that $D^{'}$ in (8.51) is not effective. In other words, we need to show, that $H^0(B^{'}, \mathcal{O}_{B'}(D^{'})) = 0$. Happily, this is easy to do. Let us assume that $D^{'}$ is an effective divisor in $B^{'}$. Furthermore, assume that the intersection of $D^{'}$ with some other effective divisor $e^{'}$ is negative. Then the divisor $D^{'} - e^{'}$ is effective. That this is true can be seen as follows. Let us write the short exact sequence

$$0 \longrightarrow \mathcal{O}_{B'}(D^{'} - e^{'}) \longrightarrow \mathcal{O}_{B'}(D^{'}) \longrightarrow \mathcal{O}_{e'}(-n) \longrightarrow 0\ . \tag{8.54}$$

Here $\mathcal{O}_{B'}(D^{'} - e^{'})$ denotes the sheaf of all sections of the line bundle $\mathcal{O}_{B'}(D^{'})$ which vanish along the curve $e^{'}$ and $\mathcal{O}_{e'}(-n)$ is a line bundle on $e^{'}$ of negative degree where $-n = D^{'} \cdot e^{'}$, $n \in \mathbb{Z}_{>0}$. Each short exact sequence of sheaves implies a long exact sequence in sheaf cohomology. The long exact sequence associated with (8.54) is given by

$$0 \to H^0\left(B^{'}, \mathcal{O}_{B'}\left(D^{'} - e^{'}\right)\right) \to H^0\left(B^{'}, \mathcal{O}_{B'}\left(D^{'}\right)\right) \to H^0\left(B^{'}, \mathcal{O}_{e'}(-n)\right) \to \cdots\ . \tag{8.55}$$

Since $\mathcal{O}_{e'}(-n)$ has negative degree, $H^0(B^{'}, \mathcal{O}_{e'}(-n)) = 0$. Hence, (8.55) implies that

$$H^0\left(B^{'}, \mathcal{O}_{B'}\left(D^{'} - e^{'}\right)\right) \cong H^0\left(B^{'}, \mathcal{O}_{B'}\left(D^{'}\right)\right). \tag{8.56}$$

Since $H^0(B^{'}, \mathcal{O}_{B'}(D^{'})) \neq 0$ by assumption, it follows that $H^0(B^{'}, \mathcal{O}_{B'}(D^{'} - e^{'})) \neq 0$. Hence, the divisor $D^{'} - e^{'}$ is effective, as claimed above. Now take $e^{'}$ to be any one of the divisors $e_i, i = 2, 4, 6$. Recall that these are a subset of the nine divisors on $B^{'}$ with self-intersection -1 discussed in section 2. Assume that $D^{'}$ is effective. Since $D^{'} \cdot e_2^{'} = -1$, where we have used the intersection numbers given in [33], the divisor $D^{'} - e_2^{'}$ has to be effective.

Furthermore, since $(D^{'} - e_2^{'}) \cdot e_4^{'} = -1$, the divisor $D^{'} - e_2^{'} - e_4^{'}$ must also be effective. Finally, using $(D^{'} - e_2^{'} - e_4^{'}) \cdot e_6^{'} = -1$, we find that the divisor

$$D^{'} - e_2^{'} - e_4^{'} - e_6^{'} = -e_9^{'} - n_1^{'} - n_2^{'} \tag{8.57}$$

is also effective. But this is clearly impossible, since $e_9^{'}, n_1^{'}$ and $n_2^{'}$ are independent effective divisors. Hence, we have shown that $D^{'}$ can not be effective, as required. The same reasoning can be applied to show that any integral multiple $nD^{'}$, for positive integer n, can not be effective. One can now Kleinmann's criterion for ampleness to conclude that there must exist an ample class $h^{'}$ such that

$$D^{'} \cdot h^{'} < 0\,. \tag{8.58}$$

Therefore, we have shown that condition (8.53) can always be satisfied. If follows that we have found vector bundles V on X which fulfill all of the numerical conditions.

Before concluding this section, let us make two final observations. First, the above argument clearly breaks down for the divisor

$$D^{'} + f^{'}\,. \tag{8.59}$$

This follows simply from the fact that $f^{'} \cdot e_i = 1$ for $i = 1, \ldots, 9$ and, hence, that $(D^{'} + f^{'}) \cdot e_i = 0$ for $i = 2, 4, 6$. It can indeed be shown [42] that $D^{'} + f^{'}$ is effective. This fact puts a severe restriction on the possible solutions of the numerical conditions, since it makes (8.44), in conjunction with (8.41) and the integral conditions (8.42) and (8.43), hard to solve. Be that as it may, there may be, in addition to (8.45), other solutions of the numerical constraints. We will not pursue them in this paper.

The second observation is of fundamental importance in our construction. Note that the argument leading to expression (8.58) required the existence of two effective divisors, $n_1^{'}$ and $n_2^{'}$, in $D^{'}$. What is the origin of these two classes? Recall that for the restricted two parameter family of rational elliptic surfaces $B^{'}$ that we are considering in this paper, $H_2(B^{'}, \mathbb{Z})^{\mathrm{inv}} \otimes \mathbb{Q}$ is generated by the invariant classes $i^{'}, f^{'}, n_1^{'} + o_2^{'}$ and $M^{'}$, where $i^{'}$ and $M^{'}$ are defined in (5.7). Note that $n_1^{'}$ enters $i^{'}$ and $M^{'}$, as well as $n_1^{'} + o_2^{'}$, whereas $n_2^{'}$ appears in $M^{'}$. Both $n_1^{'}$ and $n_2^{'}$ are irreducible components of some I_2 fibers and, hence, are effective. These two classes enter $D^{'}$ through its definition (8.51) as an element of $H_2(B^{'}, \mathbb{Z})^{\mathrm{inv}} \otimes \mathbb{Q}$. What would happen if instead of the two parameter family, we considered the less restrictive three parameter family of rational elliptic surfaces $B^{'}$? In this case, it is not too hard to show that the rank of $H_2(B^{'}, \mathbb{Z})^{\mathrm{inv}} \otimes \mathbb{Q}$ is the same, that is, four. In addition, the structure of its generators is similar to the above but with the important difference that $n_1^{'}$ and $n_2^{'}$ are now replaced by two different classes, which we denote by $l_1^{'}$ and $l_2^{'}$. These new classes are not components of I_2 fibers and, crucially, are not effective. It follows that the argument leading to expression (8.58) will no longer be valid. Hence, for generic surfaces $B^{'}$ in the three parameter family, we are unable to satisfy the necessary constraints (**C1**), (**C2**), (**C3**) and (7.13). This explains, at long last, why we have required $B^{'}$ to be in the restricted two parameter family of rational elliptic surfaces.

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To conclude this section, let us state what has been achieved so far. We have constructed rank four holomorphic vector bundles V as extensions

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{8.60}$$

of V_2 by V_1, where

$$V_1 = \pi^{'*} W_1 \otimes \pi^* L_1 \,, \qquad V_2 = \pi^{'*} W_2 \otimes \pi^* L_2 \tag{8.61}$$

are $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. The rank two vector bundles W_i, $i = 1, 2$ are taken to be locally free extensions of rank one torsion free sheaves on B,

$$0 \longrightarrow \mathcal{O}_B(a_i f) \longrightarrow W_i \longrightarrow \mathcal{O}_B(b_i f) \otimes I_{z_{k_i}} \longrightarrow 0 \;, \qquad i = 1, 2 \tag{8.62}$$

with $k_1 = 2$, $k_2 = 8$ and $a_i + b_i \in 2\mathbb{Z}$ for both $i = 1, 2$. The line bundles L_i, $i = 1, 2$ have first Chern class of the form

$$c_1(L_1) = D^{'} - \frac{1}{2}(a_1 + b_1) f^{'} \;, \qquad c_1(L_2) = -D^{'} - \frac{1}{2}(a_2 + b_2) f^{'} \tag{8.63}$$

with the divisor $D^{'}$ given by

$$D^{'} = e^{'}_2 + e^{'}_4 + e^{'}_6 - e^{'}_9 - n^{'}_1 - n^{'}_2 \,. \tag{8.64}$$

Any rank four vector bundle V in $\mathrm{Ext}^1(V_2, V_1)$ fulfills all of the numerical conditions (#**C1**), (#**C2***a*), (#**C2***b*), (#**C3**), (8.16) and (8.19). However, as stated above, demonstrating that condition (8.16) is satisfied is a necessary, but not sufficient, condition for V to be stable. Therefore, in the next section, we analyze the remaining conditions required to prove the stability of V.

9. Proof of the stability of V

In the previous section, we presented rank four holomorphic vector bundles V that satisfy all the constraints (**C1**), (**C2**) and (**C3**) required by particle physics phenomenology. In addition, we showed that these bundles also satisfy an important necessary condition for stability, namely (7.13). In this section, we will prove the stability of the rank four holomorphic vector bundles constructed in the previous section for a large class of torsion free sub-sheaves. More precisely, we will show that all torsion free sub-sheaves of V, whose restriction to the generic fiber f of $\pi : X \to B^{'}$ have negative degree, can never destabilize V. The remaining relevant torsion free sub-sheaves, those whose restriction to a generic fiber have vanishing degree, are much harder to analyze. This analysis will be carried out in [42] where it will be shown that, subject to a mild further restriction on V, these sub-sheaves also can never destabilize V. We will simply state this final restriction at the end of this section. As in section 8, nowhere in this section will we require that V be $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, only that V_1 and V_2 are. It follows that our results will apply to any element in $\mathrm{Ext}^1_X(V_2, V_1)$ and, hence, to the invariant bundles in $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ by restriction.

We begin with the observation that if F is a torsion free sub-sheaf of V, which we denote by

$$F \subset V\,, \tag{9.1}$$

then the degree of F restricted to a fiber f of the elliptic fibration $\pi : X \to B^{'}$ is non-positive. Note that we have shown in [32] using the fiber product structure $X = B \times_{\mathbb{P}^1} B^{'}$ that the fiber of $\pi : X \to B^{'}$ can be identified with the fiber of $\beta : B \to \mathbb{P}^1$. It is consistent, therefore, to denote both fibers by f. Then, we claim that

$$c_1\left(F|_f\right) \leq 0\,. \tag{9.2}$$

Also, note that

$$F|_f \subset V|_f\,. \tag{9.3}$$

The proof of (9.2) proceeds in several steps. We begin by restricting the sequence (8.60) to a smooth fiber f. Then, $V|_f$ fits into the short exact sequence

$$0 \longrightarrow V_1|_f \longrightarrow V|_f \longrightarrow V_2|_f \longrightarrow 0\,. \tag{9.4}$$

Let us analyze $V_1|_f$. First, using (8.61), note that

$$V_1|_f \cong \left(\pi^{'*}W_1 \otimes \pi^* L_1\right)|_f \cong \pi^{'*}W_1|_f \otimes \pi^* L_1|_f\,. \tag{9.5}$$

Since $\pi^* L_1$ is the pull-back of a line bundle under $\pi : X \to B^{'}$ and we choose f to be a generic fiber of π, then the restriction $\pi^* L_1|_f$ has degree zero. Furthermore, it is isomorphic to the trivial bundle $\mathcal{O}_f$ on the fiber f. That is,

$$\pi^* L_1|_f \cong \mathcal{O}_f\,. \tag{9.6}$$

Hence,

$$V_1|_f \cong \pi^{'*}W_1|_f\,. \tag{9.7}$$

Note from the identification of the fibers of X and B that

$$\pi^{'*}W_1|_f \cong W_1|_f\,. \tag{9.8}$$

If we restrict the defining sequence (8.62) of W_1 to a generic smooth fiber f which does not contain points in the ideal sheaf I_{z_2}, we obtain

$$0 \longrightarrow \mathcal{O}_f \longrightarrow W_1|_f \longrightarrow \mathcal{O}_f \longrightarrow 0\,. \tag{9.9}$$

Hence, $W_1|_f$ corresponds to an element of $\mathrm{Ext}^1_f(\mathcal{O}_f, \mathcal{O}_f)$. We can calculate this group using sheaf cohomology. Note, using (4.6), that

$$\mathrm{Ext}^1_f(\mathcal{O}_f, \mathcal{O}_f) \cong H^1(f, \mathcal{O}_f)\,. \tag{9.10}$$

Then, Serre duality on the smooth genus one curve f implies

$$H^1(f, \mathcal{O}_f) \cong H^0(f, K_f)\,, \tag{9.11}$$

where K_f denotes the canonical bundle on the torus f. Since this bundle is trivial, it has one global section. Therefore,

$$\mathrm{Ext}^1_f(\mathcal{O}_f, \mathcal{O}_f) \cong \mathbb{C} \tag{9.12}$$

and $W_1|_f$ could, in principle, correspond to a non-trivial extension. However, as we will show in [42], $W_1|_f$ must correspond to the trivial extension and, hence,

$$W_1|_f \cong \mathcal{O}_f \oplus \mathcal{O}_f \,. \tag{9.13}$$

It follows from (9.7), (9.8) and (9.13)that

$$V_1|_f \cong \mathcal{O}_f \oplus \mathcal{O}_f \,. \tag{9.14}$$

In a similar manner, one can show that $V_2|_f \cong \mathcal{O}_f \oplus \mathcal{O}_f$. Therefore, the restricted sequence (9.4) becomes

$$0 \longrightarrow \mathcal{O}_f \oplus \mathcal{O}_f \longrightarrow V|_f \longrightarrow \mathcal{O}_f \oplus \mathcal{O}_f \longrightarrow 0 \ . \tag{9.15}$$

Using this sequence, we now continue the proof of (9.2). As the next step, consider rank one sub-bundles $F \subset V$. Recall that $F|_f \subset V_f$. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_f \oplus \mathcal{O}_f \arrow[r, "u"] & V|_f \arrow[r, "v"] & \mathcal{O}_f \oplus \mathcal{O}_f \arrow[r] & 0 \\
& & F|_f \arrow[ul, dotted, "\gamma"] \arrow[u, "i"] \arrow[ur, dotted, "\lambda"'] & & \\
& & 0 \arrow[u] & &
\end{tikzcd} \tag{9.16}$$

where i is inclusion, we see that there is a composition map $\lambda = v \circ i$ such that

$$\lambda : F|_f \to \mathcal{O}_f \oplus \mathcal{O}_f \,. \tag{9.17}$$

Let us assume this map is zero, that is, $\lambda = 0$. This assumption implies that the image $i(F|_f) \subset V|_f$ maps to zero under the quotient map $v : V|_f \to \mathcal{O}_f \oplus \mathcal{O}_f$. Since (9.15) is exact, the image $i(F|_f) \subset V|_f$ has a pre-image $u^{-1}(i(F|_f))$ in $\mathcal{O}_f \oplus \mathcal{O}_f$. Hence, there exists a map $\gamma = u^{-1} \circ i$ such that

$$\gamma : F|_f \to \mathcal{O}_f \oplus \mathcal{O}_f \,. \tag{9.18}$$

Projecting the image of γ into either of these two factors gives us a map $\bar{\gamma} : F|_f \to \mathcal{O}_f$. Then, either $F|_f$ has negative degree or $F|_f \cong \mathcal{O}_f$. That this is true can easily be seen. Any holomorphic map

$$\bar{\gamma} : F|_f \to \mathcal{O}_f \tag{9.19}$$

is an element of $\mathrm{Hom}_f(F|_f, \mathcal{O}_f)$. Equivalently, since

$$\mathrm{Hom}_f(F|_f, \mathcal{O}_f) \cong H^0(f, F^*|_f \otimes \mathcal{O}_f) \,, \tag{9.20}$$

$\bar{\gamma}$ corresponds to a global section of $F^*|_f$. But if $F|_f$ has positive degree, then $F^*|_f$ has negative degree. Hence, there are no global sections of $F^*|_f$ and $\bar{\gamma}$ does not exist, in contradiction to the above result. Hence $\deg(F|_f) \leq 0$. Let us consider the case when this degree vanishes. But a line bundle of degree zero which has a global section is the trivial line bundle, hence $F|_f \cong \mathcal{O}_f$.

What happens if the map λ in (9.17) is not the zero map? We now show that, in this case, λ must to be injective, that is,

$$\ker \lambda = 0 \,. \tag{9.21}$$

To prove this, assume λ is not injective. Then, $\ker \lambda$ is a non-trivial sub-sheaf of $F|_f$ and, hence, is of rank zero or rank one. If $\ker \lambda$ has rank zero, then it would be torsion in F_f,

$$0 \to \ker \lambda \to F|_f \,. \tag{9.22}$$

But this is impossible, since we assumed $F|_f$ to be locally free and, hence, torsion free. If $\ker \lambda$ is of rank one, then the image $\lambda(F|_f)$ in $\mathcal{O}_f \oplus \mathcal{O}_f$ must be of rank zero. Therefore,

$$0 \to \lambda(F|_f) \to \mathcal{O}_f \oplus \mathcal{O}_f \tag{9.23}$$

would imply that $\lambda(F|_f)$ is a torsion sub-sheaf of $\mathcal{O}_f \oplus \mathcal{O}_f$, which is again impossible. Hence, we have shown that if λ is not the zero map then $\ker \lambda$ vanishes. It follows that

$$\lambda : F|_f \to \mathcal{O}_f \oplus \mathcal{O}_f \tag{9.24}$$

is an injection and that $F|_f$ a sub-sheaf of $\mathcal{O}_f \oplus \mathcal{O}_f$. We can now argue as above, proving that $F|_f$ has non-positive degree. The same argument works for all torsion free sub-sheaves of any rank. Therefore, we can make the important assertion that if F is any torsion free sub-sheaf of V, then the degree of $F|_f$ is non-positive and (9.2) is proven.

We will now prove the stability of V with respect to all torsion free sub-sheaves F with strictly negative degree, that is,

$$c_1(F|_f) < 0 \,. \tag{9.25}$$

More precisely, we will show that for any vector bundle V defined from sequence (8.60), there is an ample class H with respect to which each torsion free sub-sheaf satisfying (9.25) has negative slope. Since the slope of V vanishes, it follows that these sub-sheaves can never destabilize V. As discussed previously, we choose H of the form

$$H = H_0 + nH^{'} \in H^2\left(X, \mathbb{Z}\right), \tag{9.26}$$

where n is a non-negative integer,

$$H_0 = \pi^{'*} h_0 + \pi^* h_0^{'} \tag{9.27}$$

and

$$H^{'} = \pi^* h^{'} \,. \tag{9.28}$$

Here $h_0 \in H^2(B, \mathbb{Z})$ denotes an ample class in B and $h_0^{'}, h^{'} \in H^2(B^{'}, \mathbb{Z})$ are two ample classes in $B^{'}$. The motivation for choosing the ample class H to be of the form (9.26) was given in section 7 and will now prove to be essential.

Let us first consider the case of rank one torsion free sub-sheaves. As explained in the appendix, in this case, it is enough to consider all line bundles L such that

$$L \subset V\,. \tag{9.29}$$

Using the ample class H in (9.26), the slope of L is given by

$$\mu_H(L) = c_1(L)\cdot H^2 = c_1(L)\cdot\left(H_0^2 + 2nH_0\cdot H^{'} + n^2 h^{'2} f\right). \tag{9.30}$$

Using the fact that $h^{'2} > 0$, and from assumption (9.25), we have

$$c_1(L|_f) = c_1(L)\cdot f < 0\,. \tag{9.31}$$

It follows that the last term in (9.30) is negative. If one can show that

$$c_1(L)\cdot H_0^2\,, \qquad c_1(L)\cdot H_0\cdot H^{'} \tag{9.32}$$

have an upper bound for all line bundles L, then there will always exist a positive integer n_0 such that the slope of every sub-sheaf $L \subset V$ is negative for all $n \geq n_0$.

To prove that this is indeed the case, we will use a result of [45] which states that for any given rank four vector bundle V there is an filtration,

$$0 \subset F^0 \subset F^1 \subset F^2 \subset F^3 = V\,, \tag{9.33}$$

of vector bundles F^i, $i = 0, 1, 2, 3$ of rank $i+1$ such that the quotients F^j/F^{j-1}, $j = 1, 2, 3$ are torsion free rank one sheaves. This implies that these quotients are of the form

$$F^j/F^{j-1} \cong L_j \otimes I_{X_j}\,, \qquad j = 1, 2, 3\,, \tag{9.34}$$

where the L_j are line bundles and I_{X_j} are ideal sheaves on X whose sections vanish on $X_j \subset X$, where X_j are at least of codimension two. This, implies that

$$c_1(I_{X_j}) = 0\,, \qquad j = 1, 2, 3\,. \tag{9.35}$$

What is the image of L in F^2? If it is zero, then we obtain a map

$$L \to F^3/F^2 \cong L_3 \otimes I_{X_3}\,. \tag{9.36}$$

If it is non-zero, then, using arguments similar to the above, L injects into the rank two bundle F^2. One can now proceed by induction, proving that

$$L \to F^j/F^{j-1} \cong L_j \otimes I_{X_j} \tag{9.37}$$

for some fixed j. Denote this fixed value of j by J. Let us consider the map (9.37) more closely. It clearly induces a short exact sequence

$$0 \longrightarrow L \longrightarrow L_J \otimes I_{X_J} \longrightarrow Q_J \longrightarrow 0\;, \tag{9.38}$$

where we have defined

$$Q_J \equiv (L_J \otimes I_{X_J})/L \tag{9.39}$$

However, since both L and $L_J \otimes I_{X_J}$ are of rank one, Q_J is of rank zero and, hence, torsion. Under these circumstances, there always exists an effective divisor E [45] such that the quotient

$$\tilde{Q}_J \equiv (L_J \otimes I_{X_J})/(L \otimes \mathcal{O}_X(-E)) \tag{9.40}$$

is torsion free. Since the quotient $\tilde{Q}_J$ was torsion to begin with, it must vanish. Hence, we obtain

$$L \otimes \mathcal{O}_X(-E) \cong L_J \otimes I_{X_J} \tag{9.41}$$

for some effective divisor E and some fixed J. Since E and $H^{'}$ are effective and H_0 is ample, we obtain the following restrictions on the intersection numbers

$$E \cdot H_0^2 > 0\,, \qquad E \cdot H_0 \cdot H^{'} > 0\,. \tag{9.42}$$

Taking the first Chern class of each side of (9.41), and recalling from (9.35) that $c_1(I_{X_J})$ vanishes, we obtain

$$c_1(L) - E = c_1(L_J)\,. \tag{9.43}$$

Intersection both sides with H_0^2 and $H^{'} \cdot H_0$, and using (9.42), we obtain the inequalities

$$c_1(L) \cdot H_0^2 < c_1(L_J) \cdot H_0^2 \tag{9.44}$$

and

$$c_1(L) \cdot H^{'} \cdot H_0 < c_1(L_J) \cdot H^{'} \cdot H_0\,. \tag{9.45}$$

Note that L_J is fixed once we have chosen the filtration (9.36). Hence, we have proven that there indeed exists an upper bound for each of

$$c_1(L) \cdot H_0^2\,, \qquad c_1(L) \cdot H_0 \cdot H^{'} \tag{9.46}$$

for any $L \subset V$. These bounds are given respectively by

$$c_1(L_J) \cdot H_0^2\,, \qquad c_1(L_J) \cdot H_0 \cdot H^{'}\,. \tag{9.47}$$

Note that the index J depends on the chosen line bundle. For any line bundle L, we will obtain one fixed J and the bounds given in (9.47) apply. Therefore, making n sufficiently large ensures that the slope of any line bundle L with $L \subset V$ and $c_1(L|_f) < 0$ does not destabilize V.

A similar argument can be applied to any rank two and rank three torsion free sub-sheaf of V whose restriction to the generic fiber has negative degree. To see this, first note from the appendix that instead of checking the slope of all rank two torsion free sub-sheaves of V, we need only to ensure that all line bundles

$$L \subset \wedge^2 V \tag{9.48}$$

have negative slope. Here, line bundles L whose restriction to a generic fiber f have negative degree correspond to rank two sub-sheaves of V whose restriction to the generic fiber f have negative degree. However, the rank six vector bundle $\wedge^2 V$ has a filtration similar to (9.33). This filtration will again lead to upper bounds on

$$c_1(L) \cdot H_0^2\,, \qquad c_1(L) \cdot H_0 \cdot H^{'} \tag{9.49}$$

for all $L \subset \wedge^2 V$. Therefore, an argument similar to the above can be applied to prove that all such line bundles L have negative slope. The case of rank three sub-sheaves of V whose restrictions to a generic fiber have negative degree works in the same way. In this case, one can easily prove that all line bundles

$$L \subset \wedge^3 V\,, \tag{9.50}$$

whose restrictions to a generic fiber have negative degree, also have negative slope. This proves the stability of V with respect to all sub-sheaves F where $F|_f$ has strictly negative degree.

However, it much more difficult to prove that torsion free sub-sheaves F of V, with $F|_f$ of zero degree, do not destabilize V. This is beyond the range of the techniques used in this paper. Therefore, we postpone this proof to a more technical paper [42]. Here, we will simply state the results. We find that such sub-sheaves will not destabilize a generic element V of $\mathrm{Ext}^1(V_2, V_1)$ if one imposes the mild restrictions that

$$b_1 - a_1 > 0\,, \qquad b_2 - a_2 > 2\,. \tag{9.51}$$

These constraints must, therefore, be added to those given in (8.45), namely, that $a_i + b_i \in 2\mathbb{Z}, i = 1, 2$. That is, the elements of a dense, open subset of $\mathrm{Ext}^1(V_2, V_1)$ will be stable holomorphic vector bundles if the conditions (8.45) and (9.51) are satisfied.

10. Conclusions

Combining the results of the last several sections, let us state our conclusions. We construct a pair of rank two holomorphic vector bundles W_i, $i = 1, 2$ on B as locally free extensions of rank one torsion free sheaves

$$0 \longrightarrow \mathcal{O}_B(a_i f) \longrightarrow W_i \longrightarrow \mathcal{O}_B(b_i f) \otimes I_{z_{k_i}} \longrightarrow 0\ , \qquad i = 1, 2 \tag{10.1}$$

where

$$a_i + b_i \in 2\mathbb{Z}\,, \qquad b_1 - a_1 > 0\,, \qquad b_2 - a_2 > 2 \tag{10.2}$$

and

$$k_1 = 2\,, \qquad k_2 = 8\,. \tag{10.3}$$

On $B^{'}$, we consider line bundles L_i, $i = 1, 2$ with first Chern class of the form

$$c_1(L_1) = D^{'} - \frac{1}{2}(a_1 + b_1)f^{'}\ , \qquad c_1(L_2) = -D^{'} - \frac{1}{2}(a_2 + b_2)f^{'} \tag{10.4}$$

with

$$D^{'} = e_2^{'} + e_4^{'} + e_6^{'} - e_9^{'} - n_1^{'} - n_2^{'}\,. \tag{10.5}$$

From these bundles, we create a pair of rank two holomorphic vector bundles V_i, $i = 1, 2$ on X as

$$V_i = \pi'^* W_i \otimes \pi^* L_i \,. \tag{10.6}$$

Finally, we construct rank four holomorphic vector bundles V on X as extensions of V_2 by V_1. That is,

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \ . \tag{10.7}$$

The set of all such rank four bundles V is denoted by $\mathrm{Ext}^1_X(V_2, V_1)$ and satisfies [42]

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1) \geq 1 \,. \tag{10.8}$$

The choice of parameters given in (10.2) and (10.3) guarantees that there is a dense, open subset of $\mathrm{Ext}^1_X(V_2, V_1)$ each of whose elements V is stable and satisfies the standard-like model conditions (**C1**), (**C2**) and (**C3**). Furthermore, there exists a vector subspace

$$\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \subseteq \mathrm{Ext}^1_X(V_2, V_1) \tag{10.9}$$

which carries the trivial representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. This space satisfies the constraint

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}} \geq \frac{1}{4} \dim_{\mathbb{C}} \mathrm{Ext}^1_X(V_2, V_1) \tag{10.10}$$

and, hence, always contains non-trivial extensions. Each element V in $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$. Putting everything together, we conclude that there exists a dense, open subset of $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$ each of whose elements is a stable, holomorphic rank four bundle which is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ and satisfies the standard-like model conditions (**C1**), (**C2**) and (**C3**).

We close this paper by discussing the moduli space of $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, rank four stable holomorphic vector bundles V, which we denote by $\mathcal{M}(V)^{\mathrm{inv}}$. These moduli can arise from two distinct sources. First, as we have just discussed, they are associated with the dense, open subspace of vector bundles in $\mathrm{Ext}^1_X(V_2, V_1)^{\mathrm{inv}}$. In the previous section, we have given a lower bound on this part of the moduli space. However, an exact calculation of the number of such moduli is very difficult and will not be presented here. However, there is a second source of vector bundle moduli. The reader should recall from section 6 that the vector bundles V_1 and V_2 in the defining exact sequence (6.21) were each chosen to be at a specific point in their moduli space. However, varying V_1 and V_2 within their moduli space will give continuous moduli to V. The dimension of this part of the moduli space of V is much easier to compute. It will simply be the sum of the dimensions of the moduli spaces of V_1 and V_2. This is best illustrated in a specific example. Consider the specific choice of integers

$$b_1 - a_1 = 2 \,, \qquad k_1 = 2 \tag{10.11}$$

and

$$b_2 - a_2 = 4 \,, \qquad k_2 = 8 \,. \tag{10.12}$$

Note that each obey the conditions (10.2) and (10.3). From the discussion in section 4 and expressions (4.43) and (4.46), it follows that W_1 and W_2 on B have at least one and three dimensional moduli spaces respectively. This implies that the moduli spaces for π'^*W_1 and π'^*W_2 on X are at least one and three dimensional. Since B' and X are simply connected, there are no continuous moduli corresponding to the line bundles L_i, $i = 1, 2$ and their pull-backs on X. Hence,

$$\dim \mathcal{M}(V_1)^{\mathrm{inv}} \geq 1\,, \qquad \dim \mathcal{M}(V_2)^{\mathrm{inv}} \geq 3\,. \tag{10.13}$$

Therefore, from the above discussion, we conclude that that for the specific choice of integers (10.11) and (10.12)

$$\dim \mathcal{M}(V)^{\mathrm{inv}} > 4\,. \tag{10.14}$$

A similar argument can always be applied to compute a lower bound on the dimension of $\mathcal{M}(V)^{\mathrm{inv}}$ for any choice of parameters in (10.2).

All of the bundles discussed in this conclusion are invariant, but not equivariant, on X. This is a consequence of the fact that only such bundles satisfy the particle physics constraints in the context of this paper. However, it is possible that a modification of our approach could lead to equivariant bundles on X consistent with these constraints. Note that V will be equivariant if V_1 and V_2 are individually equivariant. It is not too hard to show, using (10.6), that V_i will be equivariant if 1) both W_i and L_i are equivariant or 2) neither W_i nor L_i is equivariant. Note that the solutions in this paper correspond to equivariant W_i but non-equivariant L_i. Condition 1) might be realized by adding a non-trivial bundle to the hidden sector of the theory. This may allow one to choose L_i to be equivariant. On the other hand, condition 2) might be satisfied by choosing $W_i \notin \mathrm{Ext}^1_B(\mathcal{O}_B(b_i f) \otimes I_{z_k}, \mathcal{O}_B(a_i f))^{\mathrm{inv}}$ and, therefore, not equivariant. These options, as well as others, will be discussed elsewhere.

By construction, equivariant vector bundles preserve $N = 1$ supersymmetry. However, it is not immediately clear that supersymmetry is unbroken when bundles are invariant, but not equivariant. In general, as discussed in appendix A, such bundles contain an element B in $H^2(Z, \mathbb{Z})$. For a locally non-trivial two-form, B, $N = 1$ supersymmetry would appear to be broken in heterotic theory, since $F = B \neq 0$ rather than the required $F = 0$. However, for invariant but not equivariant bundles, B must be a pure torsion element. It follows that locally $F = 0$ and $N = 1$ supersymmetry is indeed preserved. We will discuss this important point in more detail elsewhere.

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A. Invariant and equivariant vector bundles

In this paper, we construct twisted and untwisted vector bundles on non-simply connected Calabi-Yau threefolds Z. In order to do so, we consider Calabi-Yau threefolds X with a fixed point free $\mathbb{Z}_2 \times \mathbb{Z}_2$ group action. Hence we obtain a degree four cover

$$q_4 : X \to Z\,. \tag{A.1}$$

In order to obtain a vector bundle V_Z on Z, we construct vector bundles V on X which descend to bundles on Z. The descent properties of V are captured in the way V transforms under the action of the group $\mathbb{Z}_2\times\mathbb{Z}_2$. There are two possible types of behavior of V relative to this group action. These lead to a twisted or untwisted vector bundle V_Z respectively.

We begin by recalling the general notions of descent, invariance and equivariance in our context. Suppose that Y is any manifold and Γ is a finite group acting freely. Let $M \equiv Y/\Gamma$ be the quotient manifold and $q : Y \to M$ the quotient map. For a vector bundle V on Y, we say that:

- V *descends* to M if we can find a vector bundle V_M on M such that V and q^*V_M are isomorphic as bundles on Y;
- V *is* Γ*-invariant* if for every element $g \in \Gamma$ we have some isomorphism $\phi_g : V \tilde{\to} g^*V$;
- V *is* Γ*-equivariant* if there is an action of Γ on the total space of V which covers the given action of Γ on Y and is linear on the fibers of $V \to Y$.

By descent theory, one knows that in the case of a free Γ-action (which is the only case we will be looking at) the equivariance of V implies that V descends to M. In fact, when V is equivariant, the descended bundle V_M is simply the quotient $V_M = V/\Gamma$. Also, from the definitions it is clear that every Γ-equivariant V is also Γ-invariant. In general, the property of V being Γ-equivariant is strictly stronger than the property of V being Γ-invariant. To compare these two notions, it is useful to observe that an equivariant structure on V is the same thing as a choice of vector bundle isomorphisms $\phi_g : V \to g^*V$ for all $g \in \Gamma$, satisfying the extra constraint that

$$g^*(\phi_h) \circ \phi_g = \phi_{hg} \tag{A.2}$$

for any two elements $g, h \in \Gamma$.

Example A.1 Let E be an elliptic curve with an origin $e \in E$ and let $\Gamma \cong \mathbb{Z}_2$. Specifically take $\Gamma = \{1, t_{e'}\}$ for some non-trivial point $e' \in E$ of order two on E. Then Γ acts freely on E and we have a double cover map $q : E \to E'$ to the quotient $E' \equiv E/\Gamma$ which is another elliptic curve.

(i) If L is a line bundle of degree one on E, then L can not descend to E' since any line bundle on E which is a pull-back of a line bundle on E' must have even degree. In particular, L can not be equivariant for Γ. It is not hard to check that L is not even invariant. Indeed, since $\deg L = 1$ it follows that $L = \mathcal{O}_E(p)$ for some point $p \in E$. Thus $t_{e'}^* L = \mathcal{O}_E(t_{-e'}(p))$. But $t_{-e'} = t_{e'}$, since e' is of order two. Also, by the definition of the group law on E, the point $t_{e'}(p)$ is the unique point on E for which

$$\mathcal{O}_E(t_{e'}(p) - e) \cong \mathcal{O}_E(p - e) \otimes \mathcal{O}_E(e' - e) = \mathcal{O}_E\left(p + e' - 2e\right). \qquad \text{(A.3)}$$

Thus $t_{e'}^* L \cong \mathcal{O}_E(p + e - e') = L \otimes \mathcal{O}_E(e - e')$ and, hence, L is never invariant.

(ii) The same argument shows that for any line bundle L on E we have $t_{e'}^* L = L \otimes (\mathcal{O}_E(e' - e)^{\otimes \deg L})$. So, if L is any line bundle on E of even degree, then L will be automatically Γ-invariant and if L is of odd degree, then L is not Γ-invariant. In fact, it is easy to see that line bundles of even degree on E will be Γ-equivariant. Indeed, if L is such a bundle, then we have some isomorphism $\phi : L \to t_{e'}^* L$. The isomorphism $t_{e'}^*(\phi) \circ \phi : L \to L$ must be a multiplication by some constant $a \in \mathbb{C}^*$. Let $b \in \mathbb{C}^*$ be a square root of $1/a$. Then the isomorphism $b \cdot \phi : L \to t_{e'}^* L$ satisfies $t_{e'}^*(b \cdot \phi) \circ (b \cdot \phi) = \mathrm{id}_L$ and so constitutes an equivariant structure on L. Clearly, the same argument will show that any Γ-invariant vector bundle V with $H^0(E, \mathrm{End}(V)) = \mathbb{C}$ will be Γ-equivariant.

(iii) To construct vector bundles that are invariant but not equivariant, we need to work with more complicated group actions. Consider the group Λ consisting of translations by points of order two on E. In other words $\Lambda = \{1, t_{e'}, t_{e''}, t_{e'''}\}$, where e', e'' and e''' are the non-trivial points of order two on E. Now $E/\Lambda = E$ and the quotient map $q : E \to E$ is just the multiplication by two in the group law on E. Consider now a line bundle L of degree two on E. By the argument in (i), we see that $\lambda^* L \cong L$ for all $\lambda \in \Lambda$, that is, L is Λ-invariant. However L can never be Λ-equivariant since the bundles that descend to E/Λ must have a degree divisible by four.

As we saw above, a vector bundle V on Y will descend to $M = Y/\Gamma$ if and only if V is equivariant. We also saw that bundles that are invariant but not equivariant will not descend to bundles on Y/Γ. Since invariance, albeit different, is nevertheless very close to equivariance, it will be useful to understand the type of object on M which will descend from an invariant, but not equivariant, bundle on Y. This naturally leads to considering bundles on gerbes, or equivalently, twisted bundles on M.

Suppose that V is a Γ-invariant vector bundle on Y. Then we have a natural group $\mathcal{G}(V)$ associated with V defined as follows

- $\mathcal{G}(V)$ consists of all pairs (g, ϕ), where $g \in \Gamma$ and $\phi : V \to g^* V$ is a vector bundle isomorphism,
- The product of two elements $(g, \phi), (h, \psi) \in \mathcal{G}(V)$ is given by the rule

$$(g, \phi) \cdot (h, \psi) = (gh, h^*(\phi) \circ \psi). \qquad \text{(A.4)}$$

Note that the group $\mathcal{G}(V)$ fits in a short exact sequence of groups

$$1 \to \mathrm{GL}(V) \to \mathcal{G}(V) \to \Gamma \to 1\,, \qquad \text{(A.5)}$$

where $\mathrm{GL}(V)$ is the group of vector bundle automorphisms of V. Furthermore, V will be Γ-equivariant if there is a group homomorphism $\Gamma \to \mathcal{G}(V)$ splitting the sequence (A.5) and the set of all such homomorphisms is in a one-to-one correspondence with equivariant structures on V or, equivalently, with bundles on M that pull-back to V.

Now the group $\mathcal{G}(V)$ acts naturally on the total space of V in a manner compatible with the action of Γ on the base. In other words, if we use the homomorphism $\mathcal{G}(V) \to \Gamma$ to define a non-faithful action of $\mathcal{G}(V)$ on Y, then V becomes a $\mathcal{G}(V)$-equivariant bundle on Y. In particular, V descends to a vector bundle on the stack quotient $[Y/\mathcal{G}(V)]$ (see [36] for details). Since the action of $\mathcal{G}(V)$ on Y was defined so that $GL(V)$ stabilizes every point, it follows that $[Y/\mathcal{G}(V)]$ is a $\mathrm{GL}(V)$-gerbe on M.

If, in addition, V is a simple bundle, that is if $\mathrm{GL}(V) = \mathbb{C}^*$, then $[Y/\mathcal{G}(V)]$ is a $\mathbb{C}^*$-gerbe on M. Such gerbes are classified by elements in $H^2(M, C^\infty(\mathbb{C}^*))$, where $C^\infty(\mathbb{C}^*)$ denotes the sheaf of germs of complex nowhere vanishing differentiable functions on M. It is well understood in physics that elements in $H^2(M, C^\infty(\mathbb{C}^*)$ classify gauge equivalence classes of B-fields on M, so it is not surprising that the bundles on the gerbe $[Y/\mathcal{G}(V)]$ should be related to instantons on M appearing in a non-trivial B-field background. Since Γ was assumed to be finite, we can be even more precise. When V is simple, the extension (A.5) becomes the central extension

$$1 \to \mathbb{C}^* \to \mathcal{G}(V) \to \Gamma \to 1$$

and so is given by a group cohomology element $e_V \in H^2(\Gamma, \mathbb{C}^*)$. But Γ is finite and, hence, this element must come from some cohomology class $e_V^{\mathrm{fin}} \in H^2(\Gamma, \boldsymbol{\mu}_d)$, where $\boldsymbol{\mu}_d \subset \mathbb{C}^*$ is the subgroup of d-th roots of unity and d is just the least common multiple of the orders of all elements in Γ. In particular we have a sub-extension

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{G}(V) \arrow[r] & \Gamma \arrow[r] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_d \arrow[r] \arrow[u, hook] & \mathcal{G}^{\mathrm{fin}}(V) \arrow[r] \arrow[u, hook] & \Gamma \arrow[r] \arrow[u, equal] & 1
\end{tikzcd} \tag{A.6}$$

corresponding to the class e_V^{fin}. This means that the $\mathbb{C}^*$-gerbe $[Y/\mathcal{G}(V)]$ is induced from the $\boldsymbol{\mu}_d$-gerbe $[Y/\mathcal{G}^{\mathrm{fin}}(V)]$. Alternatively, this property can be expressed by saying that the class in $H^2(M, C^\infty(\mathbb{C}^*))$ representing the gerbe $[Y/\mathcal{G}(V)]$ comes from a class in $H^2(M, C^\infty(\boldsymbol{\mu}_d))$. Since $\boldsymbol{\mu}_d$ is finite, the sheaf $C^\infty(\boldsymbol{\mu}_d)$ of smooth maps from M to $\boldsymbol{\mu}_d$ is a subsheaf in the constant sheaf $\mathbb{C}^*$. In particular, the gerbe $[Y/\mathcal{G}(V)]$ comes from a flat B-field.

Vector bundles on $\mathbb{C}^*$-gerbes can be described explicitly in several different ways [36]. For the purposes of our discussion, we need a specific description of such bundles using the formalism of twisted bundles [34]. In this formalism, the data of specifying a vector bundle on a gerbe on M is replaced by the equivalent data of specifying a twisted vector bundle on M with twisting given by the equivalence class of the gerbe, which is an element in $H^2(M, C^\infty(\mathbb{C}^*))$. Concretely, if $\alpha \in H^2(M, C^\infty(\mathbb{C}^*))$ and if ${}_\alpha M$ is the corresponding $\mathbb{C}^*$-gerbe, then a vector bundle W on ${}_\alpha M$ is the same thing as an α-twisted vector bundle on M. The α-twisted vector bundles are most conveniently described via a cocycle representative of α. Choose an open covering $\{U_i\}$ of M such that in that covering α is represented by a Čech cocycle $\{\alpha_{ijk}\}$ with $\alpha_{ijk} \in H^0(U_{ijk}, C^\infty(\mathbb{C}^*))$. Then an α-twisted bundle W on M is a collection $\{\{W_i\}, \{g_{ij}\}\}$, where

- W_i is a vector bundle on U_i;
- $g_{ij} : W_{j|U_{ij}} \tilde{\to} W_{i|U_{ij}}$ is a vector bundle isomorphism;
- g_{ij} satisfy the α-twisted cocycle condition

$$g_{ij}g_{jk}g_{ki} = \alpha_{ijk} \cdot \mathrm{id} \, .$$

Going back to the question of invariance, we can now say that a simple Γ-invariant vector bundle V on Y descends to an $\alpha(V)$-twisted bundle V_M on M, where $\alpha(V) \in H^2(M, C^\infty(\mathbb{C}^*))$ is the class of the gerbe $[Y/\mathcal{G}(V)]$. However, for describing a heterotic string compactification on (M, V_M) we need to specify more than the geometric background of a twisted vector bundle. We need to understand the gauge fields on twisted bundles and their gauge symmetries. The natural framework for this is provided by the notion of a twisted connection. These twisted connections do not depend only on the smooth gerbe $\alpha(V) \in H^2(M, C^\infty(\mathbb{C}^*))$ twisting the vector bundle, but also require the extra data of a full physical B-field whose underlying gerbe is $\alpha(V)$. Recall that the standard interpretation of a B-field as a locally defined two-form gauge field is equivalent to the mathematical notion of a connection on a gerbe [35, 47, 48, 49]. As usual, we are interested only in flat B-fields, that is, flat connections on gerbes, so that we will preserve supersymmetry. A gerbe together with a flat connection fit together in a cocycle representing a class in $H^2(M, \mathbb{C}^*)$. To spell this out, we resolve the constant sheaf $\mathbb{C}^*$ by the multiplicative de Rham complex

$$\mathbb{C}^* \longrightarrow C^\infty(\mathbb{C}^*) \xrightarrow{d\log} A^1_{M,\mathbb{C}} \xrightarrow{d} A^2_{M,\mathbb{C}} \xrightarrow{d} \cdots \, . \tag{A.7}$$

Now, we have

$$H^2(M, \mathbb{C}^*) = \mathbb{H}^2 \left(M, C^\infty(\mathbb{C}^*) \xrightarrow{d\log} A^1_{M,\mathbb{C}} \xrightarrow{d} A^2_{M,\mathbb{C}} \xrightarrow{d} \cdots \right) \tag{A.8}$$

and, hence, we can represent any element in $H^2(M, \mathbb{C}^*)$ by a hyper-cohomology 2-cocycle $\{(\alpha_{ijk}, \Lambda_{ij}, B_i)\}$, where α_{ijk} is a nowhere vanishing smooth complex function on U_{ijk}, Λ_{ij} is a smooth one form on U_{ij}, and B_i is a smooth two-form on U_i. Furthermore, $\{(\alpha_{ijk}, \Lambda_{ij}, B_i)\}$ must satisfy the cocycle conditions

$$\delta\alpha = 0 \, , \qquad \delta\Lambda = d\log\alpha \, , \qquad \delta B = d\Lambda \, , \qquad dB = 0 \, , \tag{A.9}$$

where δ is the usual Čech coboundary operator.

Thus, a flat B-field $\boldsymbol{B}$ comprises the data $\{(\alpha_{ijk}, \Lambda_{ij}, B_i)\}$ up to the natural gauge transformations given by the coboundaries. Also, the $\{\alpha_{ijk}\}$ part of the B-field gives a cohomology class $\alpha \in H^2(M, C^\infty(\mathbb{C}^*))$ and so determines the gerbe underlying our B-field.

Note that in the particular case of the gerbe $[Y/\mathcal{G}(V)]$ coming from an invariant bundle V, we have a natural choice of a gauge equivalence class of flat connections since this gerbe is induced from the Deligne-Mumford gerbe $[Y/\mathcal{G}^{\text{fin}}(V)]$.

We are now ready to define a twisted connection on the twisted bundle V_M. If $\alpha(V)$ is represented by a Čech cocycle $\{\alpha_{ijk}\}$ and V_M is represented by $\{(V_M)_i, g_{ij}\}$, then an $\alpha(V)$-twisted connection on V_M is a collection of connections $\boldsymbol{\nabla} = \{\nabla_i\}$, where

- ∇_i is a connection on the vector bundle $(V_M)_i$,
- $\{\nabla_i\}$ satisfy the $\boldsymbol{B}$-twisted gauge invariance condition

$$\nabla_j = g_{ij}^* \nabla_i - \Lambda_{ij} \cdot \text{id} \ . \tag{A.10}$$

Observe next that the twisted cocycle condition on V_M becomes an ordinary cocycle condition on $\text{ad}(V_M)$. Therefore, $\text{ad}(V_M)$ is an untwisted vector bundle on M. Combined with the twisted gauge invariance condition, this gives

$$\begin{aligned} F_{\nabla_j} &= \text{ad}_{g_{ij}} F_{\nabla_i} - d\Lambda_{ij} \cdot \text{id} \\ &= \text{ad}_{g_{ij}} F_{\nabla_i} - (B_i - B_j) \cdot \text{id} \ . \end{aligned} \tag{A.11}$$

Hence, $F_{\nabla_i} - B_i$ is gauge invariant. In other words, $F_{\boldsymbol{\nabla}} - B$ is a well-defined two-form with coefficients in the untwisted vector bundle $\text{ad}(V_M)$. Therefore, for a twisted connection we can write a $\boldsymbol{B}$-corrected hermitean Yang-Mills equation

$$F_{\boldsymbol{\nabla}} - B = 0 \, . \tag{A.12}$$

Following through the construction of $\boldsymbol{B}$, we see that every solution to the ordinary hermitean Yang-Mills equation on V gives rise to a solution of the $\boldsymbol{B}$-twisted hermitean Yang-Mills equation on V_M. Therefore the twisted bundle V_M will correspond to a heterotic vacuum if and only if V is stable, or at least poly-stable, on Y.

Applying this general discussion to our original problem of constructing standard model like vacua on Z, we see that a rank four bundle V satisfying the conditions (3.8) descends to standard model vacuum V_Z on Z, if and only if V is stable and $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant. It is also instructive to observe that since $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C}^*) = \mathbb{Z}_2$, the obstruction measuring how far a $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant bundle on X is from being equivariant is two-torsion. In particular, if V is any simple invariant bundle, the bundle $V^{\otimes 2}$ is automatically equivariant. This simple observation was used in section 5.

B. Slope stability of vector bundles

On a smooth projective variety, there is a relatively simple criterion [13, 12] for ensuring that a vector bundle carries a unique connection whose field strength satisfies the hermitean Yang-Mills equation. The vector bundle must be holomorphic, that is, its transition functions have to be holomorphic and it must be stable. The concept of holomorphic vector bundles is straightforward. In this section, we define stability for such bundles. We also give explicit criteria that a holomorphic vector bundle must satisfy in order for it to be stable.

Let us start with stable holomorphic vector bundles on a smooth complex projective curve, that is, a Riemann surface C. To begin with, we must define the slope μ of a vector bundle W on a curve C. It is defined by

$$\mu(W) = \frac{c_1(W)}{\text{rank}W} \,. \tag{B.1}$$

A vector bundle W on C is said to be stable if for each vector bundle F

$$F \subset W \,, \tag{B.2}$$

satisfying

$$\text{rank}(F) < \ \text{rank}(W) \,, \tag{B.3}$$

the slope condition

$$\mu(F) < \mu(W) \tag{B.4}$$

is obeyed [45]. Having defined stability for vector bundles on a curve C, let us consider some examples. The simplest vector bundle is a holomorphic line bundle L. When is it stable? Since the rank of a line bundle is one, there is no line bundle of lower rank and, hence, for line bundles there is nothing to check, all line bundles are stable. Let us now consider rank two vector bundles W on C. In this case, we have to check the slope condition for all line bundles $F \subset W$. That is, we must show that each F satisfies

$$\mu(F) < \mu(W) \,. \tag{B.5}$$

Before proceeding to higher dimensional varieties, let us consider a specific curve. Take $C \cong \mathbb{P}^1$ and W to be any rank two vector bundle of degree zero. Then, it is not to difficult to shown [45] that W fits into the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow W \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-a) \longrightarrow 0 \tag{B.6}$$

for some integer $a \geq 0$. Therefore, W is characterized by an element of $\text{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-a), \mathcal{O}_{\mathbb{P}^1}(a))$. Happily, we can calculate $\text{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-a), \mathcal{O}_{\mathbb{P}^1}(a))$. As explained in (4.6) of section 4,

$$\text{Ext}^1_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1}(-a), \mathcal{O}_{\mathbb{P}^1}(a)\right) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a) \otimes \mathcal{O}_{\mathbb{P}^1}(a)\big) \,. \tag{B.7}$$

Using Serre duality on $\mathbb{P}^1$, we obtain

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2a) \otimes K_{\mathbb{P}^1})^* \,. \tag{B.8}$$

But $K_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. Hence, $\mathcal{O}_{\mathbb{P}^1}(-2a) \otimes K_{\mathbb{P}^1}$ has negative degree and therefore, no global sections. It follows that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a) = 0$ and there are no non-trivial extensions of $\mathcal{O}_{\mathbb{P}^1}(-a)$ by $\mathcal{O}_{\mathbb{P}^1}(a)$. We conclude that the sequence (B.6) splits and

$$W \cong \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a) \,. \tag{B.9}$$

Hence, W has a natural sub-line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$. That is,

$$\mathcal{O}_{\mathbb{P}^1}(a) \subset W \,. \tag{B.10}$$

Since $\mu(\mathcal{O}_{\mathbb{P}^1}(a)) = a$, $a \geq 0$ and $\mu(W) = 0$, W is never stable. In general, it can be shown [45] that there are no stable bundles on $\mathbb{P}^1$ of any rank.

Although it will not play a role in this paper, it is useful to discuss a somewhat weaker condition on holomorphic vector bundles called semi-stability. In the above example, assume $a = 0$. Then,

$$W \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \tag{B.11}$$

and, hence

$$\mathcal{O}_{\mathbb{P}^1} \subset W\,. \tag{B.12}$$

Since both $\mu(\mathcal{O}_{\mathbb{P}^1})$ and $\mu(W)$ vanish, it follows that

$$\mu(\mathcal{O}_{\mathbb{P}^1}) = \mu(W)\,. \tag{B.13}$$

As we show in section 9, every other sub-bundle of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ has slope smaller zero. W with these properties is called semi-stable. It can be shown [45] that all semi-stable bundles on $\mathbb{P}^1$ are of the from $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ for some a.

Having defined stability for vector bundles on a curve and having given, at least in principle, a criterion to check it, we now generalize the notion of stability to algebraic surfaces and threefolds. To begin with, we must enlarge the space of holomorphic vector bundles to the space of coherent torsion free sheaves. Coherent sheaves on projective varieties are sheaves, which have a finite resolution by a set of vector bundles. This is discussed more fully in section 4. A torsion sheaf T is a coherent sheaf whose fiber is generically zero. More precisely, there are open sets $U \subset C$, such that the only element of $T(U)$ is the zero map. For example, the skyscraper sheaf $\mathcal{O}_p$ defined in section 4 is a torsion sheaf. A torsion free sheaf F is a coherent sheaf which has no sub-sheaves which are torsion sheaves. Clearly, vector bundles, which are locally free, that is, locally isomorphic to some finite sum of the structure sheaf, are torsion free. Therefore, we have the following inclusions

$$Vect(C) \subset Coh(C)^{\mathrm{tf}} \subset Coh(C)\,, \tag{B.14}$$

were $Vect(C)$ denotes the space of all holomorphic vector bundles on C, $Coh(C)^{\mathrm{tf}}$ the space of all coherent torsion free sheaves on C and $Coh(C)$ the space of all coherent sheaves on C.

To calculate the slope of a vector bundle, or any torsion free sheaf, on a projective variety Y of dimension larger than one, we need an ample class $H \in H^2(Y, \mathbb{Z})$. A two-form H is said to be ample, if the intersection of its Poincare dual with every irreducible, effective curve $C \subset Y$ satisfies

$$H \cdot C > 0\,. \tag{B.15}$$

It can be shown that a complex manifold admits an ample class if and only if it is a projective algebraic variety. However, it can be shown, that an elliptic fibration over projective algebraic surfaces is projective. Hence, the Calabi-Yau space X that we are considering in this paper is a projective algebraic variety, and so an ample class H will exist. The slope of a vector bundle, or any torsion free sheaf, on the Calabi-Yau threefold

X is defined with respect to an ample class H as

$$\mu_H(V) = \frac{c_1(V) \cdot H^2}{\text{rank}V} \,. \tag{B.16}$$

Then, V is stable if for all sub-sheaves F of smaller rank,

$$\mu_H(F) < \mu_H(V) \,. \tag{B.17}$$

Note that, since all sub-sheaves of torsion free sheaves are torsion free as well, we can restate this, by saying that V is stable if all torsion free sub-sheaves of smaller rank satisfy the slope condition (B.16). Now observe that a sub-sheaves having rank zero implies that their generic fiber is zero, hence they are torsion sheaves. Since there are no torsion sub-sheaves of vector bundles, there are no sub-sheaves of rank zero to check. We are left by checking that all rank one, two and three torsion free sub-sheaves F of V obey

$$\mu_H(F) < 0 \,, \tag{B.18}$$

where we assume that $c_1(V) = 0$. Let us begin with rank one sub-sheaves F. Since F is torsion free, the natural map,

$$i : F \to F^{**} \,, \tag{B.19}$$

where F^{**} is the double dual of F, is an injection. This follows from the fact, that the kernel of i consist of the torsion part of F, which vanishes by assumption. Since F^{**} is of rank one, it can be shown [45] that F^{**} is locally free and, hence, is a line bundle. Dualizing the relation $F \subset V$ twice and using (B.19), we find that

$$F \subset F^{**} \subset V^{**} \cong V \,. \tag{B.20}$$

Hence, we have shown that for any rank one sub-sheaf $F \subset W$, we obtain a locally free sub-sheaf of rank one $F^{**} \subset V$ of V. Furthermore, it can be shown that $\mu(F) = \mu(F^{**})$. It follows that instead of checking the slope condition for all rank one torsion free sub-sheaves of V, one only need to check the slope condition for the subset of locally free rank one sub-sheaves. Now recall that a locally free sheaf of rank one is a line bundle L. We conclude that V will be stable with respect to rank one torsion free sub-sheaves if and only if all line bundles

$$L \subset V \tag{B.21}$$

satisfy

$$\mu(L) < 0 \,. \tag{B.22}$$

What about sub-sheaves of higher rank? Assume we have a rank two torsion free sub-sheaf F of V. This specifies an inclusion

$$0 \to F \to V \tag{B.23}$$

which, in turn, induces a mapping

$$\wedge^2 F \to \wedge^2 V \,. \tag{B.24}$$

By definition, $\wedge^2 F$ includes all sections $s_1 \otimes s_2$ of $F \otimes F$ which are of the form $s_1 \otimes s_2 = -s_2 \otimes s_1$. If F is a rank two vector bundle, then, clearly, the rank of $\wedge^2 F$ is one. This remains true for rank two torsion free sheaves. Furthermore, since F is torsion free, $\wedge^2 F$ is torsion free [46]. Therefore, by an argument as presented in section 9 to prove (9.21), the map

$$\wedge^2 F \to \wedge^2 V \tag{B.25}$$

is an inclusion and, therefore, $\wedge^2 F$ is a rank one torsion free sub-sheaf of $\wedge^2 V$. Again, that fact that $c_1(F) = c_1(\wedge^2 F)$ for rank two vector bundles [43] continues to hold for rank two torsion free sheaves and allows us to write

$$\mu(F) = \frac{c_1(F)}{2} = \frac{1}{2} c_1(\wedge^2 F) = \frac{1}{2} \mu(\wedge^2 F)\,. \tag{B.26}$$

Therefore, we only need to show that

$$\mu(F) < 0 \tag{B.27}$$

for all rank one torsion free sub-sheaves of F of $\wedge^2 V$. However, by an argument similar to that presented above, we need only check all rank one locally free sub-sheaves of $\wedge^2 V$, that is, all line bundles L

$$L \subset \wedge^2 V\,. \tag{B.28}$$

The case of rank three torsion free sub-sheaves of V can be approached in a similar manner. That is, instead, of checking that the slope of each rank three torsion free sub-sheaves $F \subset V$, such that is negative, one only need to show that all line bundles of the form

$$L \subset \wedge^3 V \tag{B.29}$$

have negative slope. Checking the slope condition

$$\mu(L) < 0 \tag{B.30}$$

for all line bundles satisfying (B.21), (B.28) and (B.29) greatly simplifies the proof of stability of a rank four vector bundle V.

References

[1] See M. Green, J. Schwarz, E. Witten, *Superstring theory, loop amplitudes, anomalies & phenomenology*, *Cambridge Monographs on Mathematical Physics*, vol. 2 (1987) and references therein.

[2] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209].

[3] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142].

[4] E. Witten, *Strong coupling expansion of Calabi-Yau compactification*, *Nucl. Phys.* **B 471** (1996) 135 [hep-th/9602070].

[5] E.I. Buchbinder, R. Donagi and B.A. Ovrut, *Vector bundle moduli superpotentials in heterotic superstrings and M-theory*, *J. High Energy Phys.* **07** (2002) 066 [hep-th/0206203].

[6] E.I. Buchbinder, R. Donagi and B.A. Ovrut, *Superpotentials for vector bundle moduli*, *Nucl. Phys.* **B 653** (2003) 400 [hep-th/0205190].

[7] E.I. Buchbinder, B.A. Ovrut, *Vacuum stability in heterotic M-theory*, to appear.

[8] M.F. Atiyah, N.J. Hitchin, V.G.'Drinfeld and Y.I.'Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[9] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[10] R. Donagi *Principal bundles on elliptic fibrations*, *Asian J. Math.* **1** (1997) 214, [alg-geom/9702002].

[11] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9709029.

[12] K. Uhlenbeck, S. Yau, *On the existence of hermitean-Yang-Mills connections in stable vector bundles*, *Commun. Pure Appl. Math.* **XXXIX** (1986) S257.

[13] S. Donaldson, *Infinite determinants, stable bundles and curvature*, *Duke Math. J.* **54** (1986) 231.

[14] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Non-perturbative vacua and particle physics in M-theory*, *J. High Energy Phys.* **05** (1999) 018 [hep-th/9811168].

[15] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Holomorphic vector bundles and non-perturbative vacua in M-theory*, *J. High Energy Phys.* **06** (1999) 034 [hep-th/9901009].

[16] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *Heterotic M-theory in five dimensions*, *Nucl. Phys.* **B 552** (1999) 246 [hep-th/9806051].

[17] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *The universe as a domain wall*, *Phys. Rev.* **D 59** (1999) 086001 [hep-th/9803235].

[18] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *Heterotic M-theory in five dimensions*, *Nucl. Phys.* **B 552** (1999) 246 [hep-th/9806051].

[19] A. Lukas, B.A. Ovrut and D. Waldram, *Gaugino condensation in M-theory on $S^1/\mathbb{Z}_2$*, *Phys. Rev.* **D 57** (1998) 7529 [hep-th/9711197].

[20] A. Lukas, B.A. Ovrut and D. Waldram, *Five-branes and supersymmetry breaking in M-theory*, *J. High Energy Phys.* **04** (1999) 009 [hep-th/9901017].

[21] R. Donagi, B.A. Ovrut and D. Waldram, *Moduli spaces of fivebranes on elliptic Calabi-Yau threefolds*, *J. High Energy Phys.* **11** (1999) 030 [hep-th/9904054].

[22] E. Lima, B.A. Ovrut, J. Park, R. Reinbacher *Non-perturbative superpotentials from membrane instantons in heterotic M-theory.*

[23] B.A. Ovrut, T. Pantev and J. Park, *Small instanton transitions in heterotic M-theory*, *J. High Energy Phys.* **05** (2000) 045 [hep-th/0001133].

[24] J. Khoury, B.A. Ovrut, P.J. Steinhardt and N. Turok, *The ekpyrotic universe: colliding branes and the origin of the hot big bang*, *Phys. Rev.* **D 64** (2001) 123522 [hep-th/0103239].

[25] R. Donagi, B. Ovrut, T. Pantev, and D. Waldram, *Spectral involutions on rational elliptic surfaces*, *Adv. Theor. Math. Phys.* **5** (2002) 499, [math.ag/00080110008011].

[26] R. Donagi, B. Ovrut, T. Pantev, and D. Waldram, *Standard-Model bundles*, *Adv. Theor. Math. Phys.* **5** (2002) 563, [math.ag/00080100008010].

[27] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard-model bundles on non-simply connected Calabi-Yau threefolds*, *J. High Energy Phys.* **08** (2001) 053 [hep-th/0008008].

[28] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard models from heterotic M-theory*, *Adv. Theor. Math. Phys.* **5** (2002) 93 [hep-th/9912208].

[29] E. Witten, *Symmetry breaking patterns in superstring models*, *Nucl. Phys.* **B 258** (1985) 75.

[30] E. Witten, private communication.

[31] R.P. Thomas, *Examples of bundles on Calabi-Yau 3-folds for string theory compactifications*, *Adv. Theor. Math. Phys.* **4** (2000) 231, [math.ag/9912179]

[32] B.A. Ovrut, T. Pantev and R. Reinbacher, *Torus-fibered Calabi-Yau threefolds with non-trivial fundamental group*, *J. High Energy Phys.* **05** (2003) 040 [hep-th/0212221].

[33] B.A. Ovrut, T. Pantev and R. Reinbacher, *Invariant homology on standard model manifolds*, hep-th/0303020.

[34] A. Căldăraru, *Derived categories of twisted sheaves on elliptic threefolds*, *J. Reine Angew. Math.* **544** (2002) 161.

[35] A. Căldăraru, S. Katz and E. Sharpe, *D-branes, B-fields and* Ext *groups*, hep-th/0302099.

[36] R. Donagi, T. Pantev, *Torus fibrations, gerbes, and duality*, math.AG/03062130306213.

[37] C. Schoen, *On fiber products of rational elliptic surfaces with section*, *Math.Z.* **197(2)** (1988) 177.

[38] R. Donagi, B. Ovrut, R. Reinbacher, *Symmetry breaking pattern by Wilson loops in heterotic M-theory*, in preparation.

[39] R. Donagi, B. Ovrut, T. Pantev, R. Reinbacher, *Standard-like models, nucleon decay and* SU(4) *instantons*, in preparation.

[40] D. Hybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, a publication of the *Max-Planck-Institute fuer Mathematik* vol. 31, 1997.

[41] R. Hartshorne, *Algebraic geometry*, Springer-Verlag New York Inc. Graduate Text in mathematics, 52, 1977.

[42] R. Donagi, B. Ovrut, T. Pantev, R. Reinbacher, *Stable vector bundles on non-simply connected Calabi-Yau spaces*, *UPR 1018-T*.

[43] M. Nakahara, *Geometry, topology and physics*, Graduate Student Series in Physics, 1990.

[44] W. Fulton, *Intersection theory*, A Series of Modern Surveys in Mathematics, vol. 2, 1997.

JHEP01(2004)022

[45] R. Friedmann, *Algebraic surfaces and holomorphic vector bundles*, Springer Verlag New York Inc., 1991.

[46] S. Kobayashi *Differential geometry of complex vector bundles*, Princton University Press, 1987.

[47] N. Hitchin *The moduli space of special lagrangian submanifolds*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 25 (1997), no. 3-4, 503–515.

[48] E.R. Sharpe, *Discrete torsion*, *Phys. Rev.* **D 68** (2003) 126003 [hep-th/0008154].

[49] E.R. Sharpe, *Discrete torsion in perturbative heterotic string theory*, *Phys. Rev.* **D 68** (2003) 126005 [hep-th/0008184].

JHEP01(2004)022